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ON THE EVALUATION OF EUROPEAN CONTINUOUS-INSTALLMENT OPTIONS

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On the evaluation of European continuous-installment options

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Abstract

This paper is concerned with the valuation of European continuous-installment options where the aim is to determine the initial premium given a constant installment payment plan. The distinctive feature of this pricing problem is the determination, along with the initial premium, of an optimal stopping boundary since the option holder has the right to stop making installment payments at any time before maturity. Given that the initial premium function of this option is governed by an inhomogeneous Black-Scholes partial differential equation, we can obtain two alternative characterizations of the European continuous-installment option pricing problem, for which no closed-form solution is available. Firstly, we formulate the pricing problem as a free boundary problem and using the integral representation method we derive integral expressions for both the initial premium and the optimal stopping boundary. Secondly, we use the linear complementarity formulation of the pricing problem for determining the initial premium and the early stopping curve implicitly with a finite difference scheme. Finally, the pricing problem is posed as an optimal stopping problem and then implemented by a Monte Carlo approach.

Key words: Installment option; free boundary problem; integral representation method

J.E.L. classification: D81, G13.

1. Introduction

Installment options are contracts where the premium (or price) is paid in installments over the lifetime of the option, rather than as a lump-sum at time of purchase, and where the holder can be allowed to lapse the contract at any payment date before maturity. In this paper we consider the class of European continuous-installment options, in which the buyer pays a small up-front premium at inception of the trade and then a constant stream of installment premiums to acquire and keep the right, but not the obligation, to buy or sell the underlying asset at the maturity date. However, the holder can choose at any time to terminate installment payments, in which case the option lapses with no further payments on either side.

Literature on installment options is quite recent. [7] and [8] derive no-arbitrage bounds for the initial premium of an installment option and study static versus dynamic hedging strategies within a Black-Scholes framework with stochastic volatility. Their analysis is restricted to European discrete-installment options, which allows for an analogy with compound options, previously considered by [10] and [23]. [3] develops a dynamic-programming procedure to price American discrete-installment options and derives some theoretical properties of the installment option contract within the geometric Brownian motion framework. This approach is applied to installment warrants, which are actively traded on the Australian Stock Exchange. [5] proposes three alternative approaches for valuing American continuous-installment options written on assets without dividends or with constant continuous dividend yield. This analysis can be applied to value installment derivatives on both non-dividend paying stocks and foreign currencies. [6] extends this pricing framework to the theoretical case of perpetual continuous-installment options. The closed-form solution obtained when the underlying asset does not pay any dividend allow him to derive some analytical properties of the initial premium and the optimal exercise and stopping boundaries for the perpetual continuous-installment call

option. [1] and [2] use a partial Laplace transform to derive an integral equation for the location of the free stopping boundary for a European continuous-installment option and study its asymptotic behavior close to expiry. [16] and [17] apply the Laplace transform approach to solve the valuation problems of American and European continuous-installment options. Using the concept of compound options, [12] derives a closed-form solution to the initial premium of a European discrete-installment option in terms of multidimensional cumulative normal distribution functions and examines the limiting case of an installment option with a continuous payment plan.

Installment options can be found embedded in other contracts, including life insurance contracts, and are also frequently used in financing capital investment projects, with some examples given in [9]. In the field of real options a meaningful model is that due to [20], in which a firm invests in a project continuously and receives no payoff until the project is complete. Although the model of [20] bears many resemblances to a European continuous-installment option, it also has some differences, notably that the project can be resumed at a later time without loss of earlier capital outlays, whereas an installment option lapses if the holder halts installment payments.

The aim of this paper is to value European continuous-installment options written on assets without dividends or with constant continuous dividend yield by using the standard Black-Scholes framework and extending the analysis developed by [5]. The valuation of these option contracts involves a free boundary problem in that the holder can choose at any time before maturity of continuing to pay installment premiums or allowing the contract to lapse, and the decision will depend upon whether the present value of the expected payoff is greater or less than the present value of the remaining payments.

The layout of the remainder of this paper is as follows. In Section 2 we present the distinctive features of the pricing problem within the standard Black-Scholes framework. In Section 3 the pricing problem is formulated as a free boundary problem, and using the integral representation method we obtain integral expressions for both the initial premium and the optimal stopping boundary, for which some analytical properties can be derived. Linear complementarity formulation of the pricing problem is posed in Section 4, along with a general framework for solving the discretized problem on a bounded computational domain by finite difference approximation schemes. In Section 5 we show how the optimal stopping problem formulation for these options can be expressed, and then implemented by a Monte Carlo approach based on the least-squares method. Numerical results are presented in Section 6, where we compare the performance of each proposed method and further analyze the convergence from discrete- to continuous-installment options. Finally, concluding remarks are given in Section 7.

2. Pricing in the Black-Scholes framework

The interest in pricing installment option contracts stems from the determination of the initial premium given the installment payment plan or vice versa. Within the standard Black-Scholes framework and following the analysis developed by [5], we value the initial premium of the European continuous-installment option once it is known the installment per unit of time. The distinctive feature of this pricing problem is the determination, along with the initial premium, of a free boundary called the *optimal stopping boundary*.

We assume the standard model for perfect capital markets, continuous trading, no-arbitrage opportunities, a constant risk-free interest rate $r \geq 0$, and an asset without dividends or with constant continuous dividend yield $\delta \geq 0$ with price process $S = (S_t)_{t \geq 0}$ governed by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (2.1)$$

where $\mu := (r - \delta)$ and $W = (W_t)_{t \geq 0}$ is a standard Wiener process under the risk-neutral probability measure. If the underlying asset is a foreign currency, δ is replaced by the foreign risk-free interest rate r_f .

Consider a continuous-installment option written on an asset whose price process S follows (2.1) and with constant installment per unit time $q \geq 0$ and plain vanilla payoff (at any time of exercise)

$$H(S_t) = \begin{cases} (S_t - K)^+, & \text{for a call option} \\ (K - S_t)^+, & \text{for a put option} \end{cases}, \quad \forall t \geq 0, \quad (2.2)$$

where $(x)^+ = \max(x, 0)$ and $K \geq 0$ is the exercise (or strike) price of the option.

It is known that the initial premium function $V_t = V(S_t, t; q)$ of a finite-lived continuous-installment option is governed by the following inhomogeneous Black-Scholes partial differential equation (PDE)

$$\frac{\partial V_t}{\partial t} + \mu S_t \frac{\partial V_t}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - rV_t = q. \quad (2.3)$$

The term $q \geq 0$ represents the continual input of cash via the installment premium: in a time period dt a constant amount $q dt$ must be paid to keep the option alive. If $q = 0$ we have the usual Black-Scholes PDE for equity options.

Independently from the type of exercise, equation (2.3) is valid only on the continuation region, that is, on the region where it is advantageous to continue paying installment premiums since the option is worth more alive than dead. For European continuous-installment options, specifically, equation (2.3) must be solved together with the appropriate boundary condition at the free boundary, which itself is unknown and must be solved for. Similarly to the free boundary problems arising in American vanilla and exotic options pricing, the condition at the free (or optimal stopping) boundary is that the initial premium function and its Delta ($\partial V_t / \partial S$) must be continuous across the boundary.

Finally, by using no-arbitrage arguments, it can be shown that there exists an equivalence of European and American continuous-installment calls written on a non-dividend paying asset, if the installment per unit time is not greater than the riskless return of investing the strike price, i.e., if $q \leq rK$. Under these circumstances, since the additional feature of the early exercise privilege is worthless, European and American continuous-installment calls are priced the same, as well as it happens for European and American vanilla calls when the underlying asset does not pay any dividends.

3. Free boundary problem formulation

Consider a European continuous-installment option on an asset whose price process S follows (2.1) with exercise (or maturity) date T , constant installment per unit time $q \geq 0$ and plain vanilla payoff $H(S_T)$.

Let $c_t = c(S_t, t; q)$ and $p_t = p(S_t, t; q)$ be the initial premium functions of a European continuous-installment call and put option respectively, defined on the same domain $\mathcal{D} \equiv \{(S_t, t) \in [0, T] \times [0, \infty)\}$. For each time $t \in [0, T]$, there exists a lower critical asset price $A_t = A(t; q)$, with $0 < A_t \leq K$ for all $t \geq 0$, below which it is optimal to terminate payments by stopping the call option. Similarly, for each time t there must be an upper critical asset price $G_t = G(t; q)$, with $K \leq G_t < \infty$ for all $t \geq 0$, above which it is advantageous to terminate payments by stopping the put option. According to these lower and upper critical asset prices, the initial premium functions c_t and p_t satisfy respectively the following conditions

$$c_t = 0, \quad \text{if } S_t \in [0, A_t], \quad (3.1)$$

$$c_t > 0, \quad \text{if } S_t \in (A_t, \infty), \quad (3.2)$$

and

$$p_t > 0, \quad \text{if } S_t \in [0, G_t], \quad (3.3)$$

$$p_t = 0, \quad \text{if } S_t \in [G_t, \infty). \quad (3.4)$$

The *optimal stopping boundaries*, which are the time paths of lower and upper critical asset prices A_t and G_t , for $t \in [0, T]$, divide the domain \mathcal{D} into a *stopping region*

$$\mathcal{S} = \{(S_t, t) \in [0, A_t] \times [0, T]\}, \quad \text{for a call option,}$$

$$\widetilde{\mathcal{S}} = \{(S_t, t) \in [G_t, \infty) \times [0, T]\}, \quad \text{for a put option,}$$

and a *continuation region*

$$\mathcal{C} = \{(S_t, t) \in (A_t, \infty) \times [0, T]\}, \quad \text{for a call option,}$$

$$\widetilde{\mathcal{C}} = \{(S_t, t) \in [0, G_t] \times [0, T]\}, \quad \text{for a put option.}$$

To ensure that the fundamental constraints $c_t \geq 0$ and $p_t \geq 0$ are satisfied in the domain \mathcal{D} , equations (3.1) and (3.4) impose that, in the stopping region, the initial premiums c_t and p_t , respectively, are equal to zero. By contrast, the inequalities expressed in (3.2) and (3.3) show that, in the continuation region, it is advantageous to continue paying the installment premiums since, respectively, call and put are worth more alive than dead. The call and put initial premiums are given respectively by (3.1) and (3.4) if the asset price starts in the stopping region, so we assume that call and put are alive at the time $t \geq 0$ of entering into the option contract, i.e., $S_t \in (A_t, \infty)$ and $S_t \in [0, G_t)$, respectively.

The initial premium function c_t of the European continuous-installment call satisfies the inhomogeneous Black-Scholes PDE (2.3) in C , that is,

$$\frac{\partial c_t}{\partial t} + \mu S_t \frac{\partial c_t}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 c_t}{\partial S^2} - r c_t = q, \quad \text{on } C. \quad (3.5)$$

Following the analysis of [21] and [22], we determine that c_t and the stopping boundary A_t jointly solve a free boundary problem consisting of (3.5) subject to the following final and boundary conditions

$$c_T = H(S_T), \quad 0 \leq S_T < \infty, \quad (3.6)$$

$$\lim_{S_t \downarrow A_t} c_t = 0, \quad 0 \leq t < T, \quad (3.7)$$

$$\lim_{S_t \downarrow A_t} \frac{\partial c_t}{\partial S} = 0, \quad 0 \leq t < T. \quad (3.8)$$

Similarly, the initial premium function p_t of the European continuous-installment put satisfies the inhomogeneous Black-Scholes PDE (2.3) in \tilde{C} , that is,

$$\frac{\partial p_t}{\partial t} + \mu S_t \frac{\partial p_t}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 p_t}{\partial S^2} - r p_t = q, \quad \text{on } \tilde{C}, \quad (3.9)$$

subject to the following final and boundary conditions

$$p_T = H(S_T), \quad 0 \leq S_T < \infty, \quad (3.10)$$

$$\lim_{S_t \uparrow G_t} p_t = 0, \quad 0 \leq t < T, \quad (3.11)$$

$$\lim_{S_t \uparrow G_t} \frac{\partial p_t}{\partial S} = 0, \quad 0 \leq t < T. \quad (3.12)$$

The *value-matching conditions* (3.7) and (3.11) and the *high-contact conditions* (3.8) and (3.12) imply, respectively, that the initial premium functions c_t and p_t and the slopes $\partial c_t / \partial S$ and $\partial p_t / \partial S$ are continuous across the respective optimal stopping boundary. The two pairs of equations (3.7-3.8) and (3.11-3.12) are jointly referred to as *smooth fit conditions* and ensure the optimality of the early stopping policy.

3.1. Solving the free boundary problem with the integral representation method

To solve the free boundary problems (3.5–3.8) and (3.9–3.12) we adopt the integral representation method introduced in [4], [13] and [15].

Proposition 3.1. *If the underlying asset price S follows a lognormal diffusion process, then the initial premium functions c_t and p_t , for $t \in [0, T]$, of the European call and put options with a constant installment per unit time $q \geq 0$, exercise date T and plain vanilla payoff $H(S_T)$ are defined respectively by*

$$c_t = c_t^{\text{BS}} - q \int_t^T e^{-r(s-t)} N(d_2(S_t, A_s, s-t)) ds, \quad (3.13)$$

$$p_t = p_t^{\text{BS}} - q \int_t^T e^{-r(s-t)} N(-d_2(S_t, G_s, s-t)) ds, \quad (3.14)$$

where

$$d_1(x, y, \tau) = \frac{\ln\left(\frac{x}{y}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad (3.15)$$

$$d_2(x, y, \tau) = d_1(x, y, \tau) - \sigma\sqrt{\tau}, \quad (3.16)$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}v^2} dv,$$

and $c_t^{\text{BS}} = c^{\text{BS}}(S_t; K, T-t)$ and $p_t^{\text{BS}} = p^{\text{BS}}(S_t; K, T-t)$ are the Black-Scholes European call and put pricing formulas, respectively. Furthermore, the optimal stopping boundaries A_t and G_t , for $t \in [0, T]$, are implicitly defined by the following integral equations

$$0 = c^{\text{BS}}(A_t; K, T-t) - q \int_t^T e^{-r(s-t)} N(d_2(A_t, A_s, s-t)) ds, \quad (3.17)$$

$$0 = p^{\text{BS}}(G_t; K, T-t) - q \int_t^T e^{-r(s-t)} N(-d_2(G_t, G_s, s-t)) ds. \quad (3.18)$$

Proof. We solve the free boundary problem (3.5–3.8) in order to obtain the integral representation for the initial premium and the optimal stopping boundary of the European continuous-installment call. By applying the same results, the solution to the free boundary problem (3.9–3.12) for the European continuous-installment put is achieved.

Let $Z(S_{t'}, t') := e^{-r(t'-t)} c_{t'}$ be the time t discounted initial premium function of the European continuous-installment call, defined in the domain \mathcal{D} . In this domain, the function $Z(S_{t'}, t')$ inherits the properties of the initial premium function $c_{t'}$, i.e., it is a convex function in $S_{t'}$ for all t' , continuously differentiable in t' for all $S_{t'}$, and a.e. twice continuously differentiable in $S_{t'}$ for all t' . Applying Itô's Lemma to $Z(S_{t'}, t')$ yields

$$Z(S_T, T) = Z(S_t, t) + \int_t^T \frac{\partial Z(S_{t'}, t')}{\partial S} dS_{t'} + \int_t^T \left(\frac{\sigma^2}{2} S_{t'}^2 \frac{\partial^2 Z(S_{t'}, t')}{\partial S^2} + \frac{\partial Z(S_{t'}, t')}{\partial t'} \right) dt'.$$

In terms of $c_{t'}$ this means

$$e^{-r(T-t)} c_T = c_t + \int_t^T e^{-r(t'-t)} \frac{\partial c_{t'}}{\partial S} dS_{t'} + \int_t^T e^{-r(t'-t)} \left(\frac{\sigma^2}{2} S_{t'}^2 \frac{\partial^2 c_{t'}}{\partial S^2} - r c_{t'} + \frac{\partial c_{t'}}{\partial t'} \right) dt'.$$

From (3.6) we know that $c_T = (S_T - K)^+$, and decomposing the initial premium into $c_{t'} = \mathbf{1}_{\{S_{t'} \leq A_{t'}\}} \cdot 0 + \mathbf{1}_{\{S_{t'} > A_{t'}\}} \cdot c_{t'}$, we have

$$\begin{aligned} e^{-r(T-t)} (S_T - K)^+ &= c_t + \int_t^T e^{-r(t'-t)} \left(\mathbf{1}_{\{S_{t'} > A_{t'}\}} \frac{\partial c_{t'}}{\partial S} \right) (\mu S_{t'} dt' + \sigma S_{t'} dW_{t'}) \\ &\quad + \int_t^T e^{-r(t'-t)} \mathbf{1}_{\{S_{t'} > A_{t'}\}} \left(\frac{\sigma^2}{2} S_{t'}^2 \frac{\partial^2 c_{t'}}{\partial S^2} - r c_{t'} + \frac{\partial c_{t'}}{\partial t'} \right) dt'. \end{aligned}$$

On the continuation region \mathcal{C} , the initial premium function $c_{t'}$ satisfies the inhomogeneous Black-Scholes PDE (3.5), so the terms multiplying $\mathbf{1}_{\{S_{t'} > A_{t'}\}}$ sum to q . Using this and taking expectations under the risk-neutral probability measure, yields

$$c_t^{\text{BS}} := \mathbb{E} \left[e^{-r(T-t)} (S_T - K)^+ \right] = c_t + q \int_t^T e^{-r(t'-t)} N(d_2(S_t, A_{t'}, t'-t)) dt'.$$

By rearranging terms and setting $s = t'$, we obtain the integral equation (3.13) for the initial premium c_t of the European continuous-installment call. Finally, applying the boundary condition (3.7), we obtain the integral equation (3.17) satisfied by the optimal stopping boundary A_t . \square

Equation (3.13) expresses the initial premium of a European continuous-installment call as the sum of the corresponding European vanilla call value and the expected present value of installment payments along the free boundary. Equation (3.17) reflects the fact that the initial premium of a European continuous-installment call at the time of optimal stopping is equal to the option payoff, which is zero. This suggests that the initial premium of a European continuous-installment call should be computed in two steps. In the first, equation (3.17) is solved for A_t . Given the optimal stopping boundary, equation (3.13) is solved next. Unfortunately, direct solutions for the integral equations (3.13) and (3.17) are not possible. According to [18], these are nonlinear convolution Volterra integral equations with a weakly-singular kernel and can only be solved numerically.

3.2. Properties of the optimal stopping boundary

Intuitively, the early stopping decision is made by the option holder if the present value of the expected payoff is less than the present value of the remaining installment payments. By allowing the contract to lapse since continuous payments are halted, the option holder receives as benefit the present value of the remaining premiums to be paid until maturity. The cost associated with early stopping is that the option holder has to forgo, according to the type of option, either interest to be earned by investing the exercise price (for call option) or dividend paid by the underlying asset (for put option), and may regret early stopping if the option becomes worthless later on because the asset price, respectively, either decreases or increases substantially. This trade-off is reflected in the free (or optimal stopping) boundary representing, for each time $t \in [0, T]$, the critical asset price at which it is indifferent between terminating and continuing to make installments payments. In order to improve our understanding of the European continuous-installment option valuation problem, let us analyze the main properties of the free boundaries A_t and G_t of call and put, respectively.

Setting $\tau - \xi = s - t$ and $\tau := T - t$, integral equations (3.13) and (3.14) can be rewritten as

$$\begin{aligned} c_t &= c^{\text{BS}}(S_t; K, \tau) - q \int_0^\tau e^{-r(\tau-\xi)} N(d_2(S_t, a_\xi, \tau - \xi)) d\xi, \\ p_t &= p^{\text{BS}}(S_t; K, \tau) - q \int_0^\tau e^{-r(\tau-\xi)} N(-d_2(S_t, g_\xi, \tau - \xi)) d\xi, \end{aligned}$$

where $a_\xi = a(\xi; q) := A(T - \xi; q)$ and $g_\xi = g(\xi; q) := G(T - \xi; q)$. Using the above integral representations, the optimal stopping boundaries a_τ and g_τ as functions of time to maturity $\tau \in (0, T]$ are implicitly defined by

$$0 = c^{\text{BS}}(a_\tau; K, \tau) - q \int_0^\tau e^{-r(\tau-\xi)} N(d_2(a_\tau, a_\xi, \tau - \xi)) d\xi, \quad (3.19)$$

$$0 = p^{\text{BS}}(g_\tau; K, \tau) - q \int_0^\tau e^{-r(\tau-\xi)} N(-d_2(g_\tau, g_\xi, \tau - \xi)) d\xi. \quad (3.20)$$

The optimal stopping boundaries $a(\tau; q)$ and $g(\tau; q)$ for European continuous-installment call and put, respectively, have the following properties.

Proposition 3.2. *If the underlying asset price S follows a stochastic process with continuous sample paths and the risk-free interest rate r is a positive constant, then a_τ and g_τ are continuous functions of time to maturity τ .*

Proof. The intuition behind continuity property of the optimal exercise boundary for American vanilla options as it has been proved in [25], can easily be extended to the optimal stopping boundary for European continuous-installment options. Therefore, from the assumption of continuity of the underlying asset price's sample paths and the definitions of the optimal stopping boundaries for call and put options derive continuity of a_τ and g_τ , respectively. If a_τ and g_τ were discontinuous, it would be possible for the underlying asset price to be, respectively, below a_τ and above g_τ without crossing them, contradicting the definitions of the optimal stopping boundaries. \square

It is interesting to note that a monotony property does not hold for the optimal stopping boundaries a_τ and g_τ of the European continuous-installment call and put, respectively. This is due to the fact that it is not possible to sign the time derivatives of a_τ and g_τ . In order to derive the relationship between the optimal stopping boundary and the time to maturity, let us differentiate both sides of integral equations (3.19) and (3.20) with respect to τ

$$0 = \frac{\partial}{\partial \tau}(c^{\text{BS}}(a_\tau; K, \tau)) - q \frac{\partial}{\partial \tau} \left(\int_0^\tau e^{-r(\tau-\xi)} N(d_2(a_\tau, a_\xi, \tau - \xi)) d\xi \right),$$

$$0 = \frac{\partial}{\partial \tau}(p^{\text{BS}}(g_\tau; K, \tau)) - q \frac{\partial}{\partial \tau} \left(\int_0^\tau e^{-r(\tau-\xi)} N(-d_2(g_\tau, g_\xi, \tau - \xi)) d\xi \right).$$

Using the put-call parity relation $p^{\text{BS}}(S_i; K, \tau) - c^{\text{BS}}(S_i; K, \tau) = Ke^{-r\tau} - S_i e^{-\delta\tau}$ for European vanilla options and rearranging equation (3.15) to find that either $x = y e^{d_1(x,y,\tau)\sigma\sqrt{\tau} - (\mu + \frac{1}{2}\sigma^2)\tau}$ or $y = x e^{-d_1(x,y,\tau)\sigma\sqrt{\tau} + (\mu + \frac{1}{2}\sigma^2)\tau}$, we obtain

$$0 = \Theta_c^{\text{BS}}(a_\tau; \tau) - q \left[\frac{1}{2} - r \int_0^\tau e^{-r(\tau-\xi)} N(d_2(a_\tau, a_\xi, \tau - \xi)) d\xi + \int_0^\tau e^{-r(\tau-\xi)} n(d_2(a_\tau, a_\xi, \tau - \xi)) \frac{\partial}{\partial \tau} (d_2(a_\tau, a_\xi, \tau - \xi)) d\xi \right],$$

$$0 = \Theta_p^{\text{BS}}(g_\tau; \tau) - q \left[\frac{1}{2} - r \int_0^\tau e^{-r(\tau-\xi)} N(-d_2(g_\tau, g_\xi, \tau - \xi)) d\xi - \int_0^\tau e^{-r(\tau-\xi)} n(-d_2(g_\tau, g_\xi, \tau - \xi)) \frac{\partial}{\partial \tau} (d_2(g_\tau, g_\xi, \tau - \xi)) d\xi \right],$$

where

$$n(z) \equiv \frac{\partial N(z)}{\partial z} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2},$$

and

$$\Theta_c^{\text{BS}}(a_\tau; \tau) = \frac{\partial c^{\text{BS}}}{\partial \tau} = -\frac{\partial c^{\text{BS}}}{\partial t}$$

$$:= a'_\tau e^{-\delta\tau} N(d_1(a_\tau, K, \tau)) + a_\tau \frac{e^{-\delta\tau} \sigma n(d_1(a_\tau, K, \tau))}{2\sqrt{\tau}} - \delta a_\tau e^{-\delta\tau} N(d_1(a_\tau, K, \tau)) + rKe^{-r\tau} N(d_2(a_\tau, K, \tau)),$$

$$\Theta_p^{\text{BS}}(g_\tau; \tau) = \frac{\partial p^{\text{BS}}}{\partial \tau} = -\frac{\partial p^{\text{BS}}}{\partial t}$$

$$:= -g'_\tau e^{-\delta\tau} N(-d_1(g_\tau, K, \tau)) + g_\tau \frac{e^{-\delta\tau} \sigma n(d_1(g_\tau, K, \tau))}{2\sqrt{\tau}} + \delta g_\tau e^{-\delta\tau} N(-d_1(g_\tau, K, \tau)) - rKe^{-r\tau} N(-d_2(g_\tau, K, \tau)),$$

are the time derivatives of the European vanilla call and put prices, respectively, which are usually referred to in the literature as *thetas*¹, and here expressed as functions of time-dependent critical asset prices a_τ and g_τ and time to maturity τ . Substituting the expressions for $\Theta_c^{\text{BS}}(a_\tau; \tau)$ and $\Theta_p^{\text{BS}}(g_\tau; \tau)$ into the above equations and solving for a'_τ and g'_τ , respectively, yields

$$a'_\tau = \frac{da(\tau; q)}{d\tau} = -\frac{dA(t; q)}{dt} = - \left[a_\tau \left(\frac{\sigma n(d_1(a_\tau, K, \tau))}{2\sqrt{\tau} N(d_1(a_\tau, K, \tau))} - \delta \right) + rKe^{-\mu\tau} \frac{N(d_2(a_\tau, K, \tau))}{N(d_1(a_\tau, K, \tau))} \right]$$

$$+ \frac{q}{N(d_1(a_\tau, K, \tau))} \left[\frac{e^{\delta\tau}}{2} - re^{-\mu\tau} \int_0^\tau e^{r\xi} N(d_2(a_\tau, a_\xi, \tau - \xi)) d\xi \right]$$

$$+ a_\tau \int_0^\tau e^{\delta\xi} \frac{n(d_1(a_\tau, a_\xi, \tau))}{a_\xi} \frac{\partial}{\partial \tau} (d_2(a_\tau, a_\xi, \tau - \xi)) d\xi, \quad (3.21)$$

¹In the comparative statics analysis of the Black-Scholes option pricing formulas, the thetas measure the changes in option value with respect to a change in time variable and they are used to estimate how fast call and put options will lose value with the passage of time.

$$\begin{aligned}
g'_\tau = \frac{dg(\tau; q)}{d\tau} = -\frac{dG(t; q)}{dt} &= \left[g_\tau \left(\frac{\sigma n(d_1(g_\tau, K, \tau))}{2\sqrt{\tau} N(-d_1(g_\tau, K, \tau))} + \delta \right) - rKe^{-\mu\tau} \frac{N(-d_2(g_\tau, K, \tau))}{N(-d_1(g_\tau, K, \tau))} \right] \\
&\quad - \frac{q}{N(-d_1(g_\tau, K, \tau))} \left[\frac{e^{\delta\tau}}{2} - re^{-\mu\tau} \int_0^\tau e^{\tau\xi} N(-d_2(g_\tau, g_\xi, \tau - \xi)) d\xi \right. \\
&\quad \left. - g_\tau \int_0^\tau e^{\delta\xi} \frac{n(d_1(g_\tau, g_\xi, \tau))}{g_\xi} \frac{\partial}{\partial\tau} (d_2(g_\tau, g_\xi, \tau - \xi)) \right]. \tag{3.22}
\end{aligned}$$

It is not possible to sign a'_τ and g'_τ because the time derivatives may be positive or negative depending on the relative magnitudes of the two bracketed terms with opposite signs in equations (3.21) and (3.22), respectively.

We now examine the limiting behavior of the optimal stopping boundaries at expiry.

Proposition 3.3. *If the underlying asset price S follows a lognormal diffusion process and the risk-free interest rate r is a positive constant, then*

$$a(0^+) := \lim_{\tau \rightarrow 0^+} a(\tau; q) = K, \tag{3.23}$$

$$g(0^+) := \lim_{\tau \rightarrow 0^+} g(\tau; q) = K. \tag{3.24}$$

Proof. Factorizing and rearranging terms, integral equations (3.19) and (3.20) can be written as

$$\frac{a_\tau}{K} = \left(e^{-r\tau} N(d_2(a_\tau, K, \tau)) + \frac{q}{K} \int_0^\tau e^{-r(\tau-\xi)} N(d_2(a_\tau, a_\xi, \tau - \xi)) d\xi \right) \left(e^{-\delta\tau} N(d_1(a_\tau, K, \tau)) \right)^{-1}, \tag{3.25}$$

$$\frac{g_\tau}{K} = \left(e^{-r\tau} N(-d_2(g_\tau, K, \tau)) - \frac{q}{K} \int_0^\tau e^{-r(\tau-\xi)} N(-d_2(g_\tau, g_\xi, \tau - \xi)) d\xi \right) \left(e^{-\delta\tau} N(-d_1(g_\tau, K, \tau)) \right)^{-1}. \tag{3.26}$$

To find the values of a_τ and g_τ at expiry, we take the limit of equations (3.25) and (3.26) as τ tends to 0^+ . Since $0 < a_\tau \leq K$ and $K \leq g_\tau < \infty$, for all $\tau \in (0, T]$, there are two possibilities with respect to the limiting behavior of $d_1(\cdot, \cdot, \cdot)$ and $d_2(\cdot, \cdot, \cdot)$, that is,

$$\lim_{\tau \rightarrow 0^+} d_1(a_\tau, K, \tau) = \lim_{\tau \rightarrow 0^+} d_2(a_\tau, K, \tau) = \lim_{\tau \rightarrow 0^+} \frac{\ln \frac{a_\tau}{K}}{\sigma \sqrt{\tau}} = \begin{cases} -\infty, & a(0^+) < K \\ 0, & a(0^+) = K \end{cases},$$

and

$$\lim_{\tau \rightarrow 0^+} d_1(g_\tau, K, \tau) = \lim_{\tau \rightarrow 0^+} d_2(g_\tau, K, \tau) = \lim_{\tau \rightarrow 0^+} \frac{\ln \frac{g_\tau}{K}}{\sigma \sqrt{\tau}} = \begin{cases} 0, & g(0^+) = K \\ \infty, & g(0^+) > K \end{cases}.$$

However, we can prove by contradiction that both case $a(0^+) < K$ and case $g(0^+) > K$ are not possible. Assuming that $a(0^+) < K < g(0^+)$ and using the above results, we obtain that the limit as τ tends to 0^+ of equations (3.25) and (3.26) is of the indeterminate form of type $\frac{0}{0}$. Application of L'Hôpital's rule in evaluating the limit of the right-hand side of equations (3.25) and (3.26), yields

$$\lim_{\tau \rightarrow 0^+} \frac{a_\tau}{K} = \frac{\frac{q}{K} N(0)}{-N(-\infty)} = -\infty, \quad \lim_{\tau \rightarrow 0^+} \frac{g_\tau}{K} = \frac{-\frac{q}{K} N(0)}{-N(-\infty)} = \infty.$$

which is a contradiction. Then, there is only one possibility with respect to the limiting behavior of $d_1(\cdot, \cdot, \cdot)$ and $d_2(\cdot, \cdot, \cdot)$. Combining this result and taking the limit of equations (3.25) and (3.26) as τ tends to 0^+ , we obtain

$$\lim_{\tau \rightarrow 0^+} \frac{a_\tau}{K} = 1 \quad \text{and} \quad \lim_{\tau \rightarrow 0^+} \frac{g_\tau}{K} = 1,$$

which are equations (3.23) and (3.24), respectively. \square

It is easy to show that the optimal stopping boundaries $a(\tau; q)$ and $g(\tau; q)$ are, respectively, strictly increasing and decreasing functions of installment per unit time q for all $\tau > 0$. Therefore, by differentiating equations (3.25) and (3.26) with respect to q , we obtain

$$\begin{aligned}\frac{da(\tau; q)}{dq} &= \frac{e^{\delta\tau}}{N(d_1(a_\tau, K, \tau))} \int_0^\tau e^{-r(\tau-\xi)} N(d_2(a_\tau, a_\xi, \tau - \xi)) d\xi > 0, \\ \frac{dg(\tau; q)}{dq} &= -\frac{e^{\delta\tau}}{N(-d_1(g_\tau, K, \tau))} \int_0^\tau e^{-r(\tau-\xi)} N(-d_2(g_\tau, g_\xi, \tau - \xi)) d\xi < 0.\end{aligned}$$

3.3. Implementation of the valuation formulas by the Multipiece Exponential Function method

Once the integral equation defining the early stopping curve is solved, the computation of the initial premium simply implies numerical integration. Unfortunately, this integral equation cannot be solved explicitly and then an approximating method to the optimal stopping boundary is needed. The first numerical method we propose to price European continuous-installment options is to extend the Multipiece Exponential Function (MEF) method introduced by [14].

Let us divide the interval $[t, T]$ into M equal time intervals by the partition $\{t_i = t + i(T-t)/M, i = 0, 1, 2, \dots, M\}$. Let \tilde{c}_t^M , for $t \in [0, T]$, denote the approximated initial premium of a European continuous-installment call corresponding to approximating the optimal stopping boundary A_s , for $s \in [t, T]$, by an M -piece exponential function $\{A_{Mj} e^{a_{Mj}(T-t_j)}, j = 1, 2, \dots, M\}$. Then \tilde{c}_t^M is given by

$$\tilde{c}_t^M = \begin{cases} 0, & \text{if } S_t \leq A_{MM} \\ c(M, S_t, A, a, \phi, \nu, T-t), & \text{if } S_t > A_{MM} \end{cases}, \quad (3.27)$$

where

$$c(j, x, A, a, \phi, \nu, t_j - t) = c^{\text{BS}}(x; K, t_j - t) - q \sum_{i=1}^j I(t_{i-1}, t_i, x, A_{M, j-i+1}, a_{M, j-i+1}, -1, r).$$

To determine both the bases A_{Mj} and the exponents a_{Mj} , for $j = 1, 2, \dots, M$, of the M -piece exponential function, we apply the value-matching and high-contact conditions (3.7–3.8) at each time step t_j . This yields

$$\begin{cases} c(j, A_{Mj} e^{a_{Mj}(T-t_j)}, A, a, \phi, \nu, t_j - t) = 0 \\ c_x(j, A_{Mj} e^{a_{Mj}(T-t_j)}, A, a, \phi, \nu, t_j - t) = 0 \end{cases}, \quad (3.28)$$

where

$$c_x(j, x, A, a, \phi, \nu, t_j - t) = e^{-\delta(t_j-t)} N(d_1(x, K, t_j - t)) - q \sum_{i=1}^j I_x(t_{i-1}, t_i, x, A_{M, j-i+1}, a_{M, j-i+1}, -1, r).$$

The functions $I(t_{i-1}, t_i, x, y, z, \phi, \nu)$ and $I_x(t_{i-1}, t_i, x, y, z, \phi, \nu)$ are defined, respectively, by²

$$\begin{aligned}I(t_{i-1}, t_i, x, y, z, \phi, \nu) &= \frac{1}{\nu} \left(e^{-\nu t_{i-1}} N\left(z_1 \sqrt{t_{i-1}} + \frac{z_2}{\sqrt{t_{i-1}}}\right) - e^{-\nu t_i} N\left(z_1 \sqrt{t_i} + \frac{z_2}{\sqrt{t_i}}\right) \right) \\ &\quad + \frac{1}{2\nu} \left(\frac{z_1}{z_3} + 1 \right) e^{z_2(z_3 - z_1)} \left(N\left(z_3 \sqrt{t_i} + \frac{z_2}{\sqrt{t_i}}\right) - N\left(z_3 \sqrt{t_{i-1}} + \frac{z_2}{\sqrt{t_{i-1}}}\right) \right) \\ &\quad + \frac{1}{2\nu} \left(\frac{z_1}{z_3} - 1 \right) e^{-z_2(z_3 + z_1)} \left(N\left(z_3 \sqrt{t_i} - \frac{z_2}{\sqrt{t_i}}\right) - N\left(z_3 \sqrt{t_{i-1}} - \frac{z_2}{\sqrt{t_{i-1}}}\right) \right),\end{aligned}$$

²See [5] for the derivation of these functions.

$$\begin{aligned}
I_x(t_{i-1}, t_i, x, y, z, \phi, \nu) &= \frac{1}{\nu} \left(\frac{e^{-\nu t_{i-1}}}{\sqrt{t_{i-1}}} n\left(z_1 \sqrt{t_{i-1}} + \frac{z_2}{\sqrt{t_{i-1}}}\right) - \frac{e^{-\nu t_i}}{\sqrt{t_i}} n\left(z_1 \sqrt{t_i} + \frac{z_2}{\sqrt{t_i}}\right) \right) \frac{1}{\sigma x} \\
&+ \frac{1}{2\nu} \left(\frac{z_1 + z_3}{z_3} \right) \left[e^{z_2(z_3 - z_1)} (z_3 - z_1) \left(N\left(z_3 \sqrt{t_i} + \frac{z_2}{\sqrt{t_i}}\right) - N\left(z_3 \sqrt{t_{i-1}} + \frac{z_2}{\sqrt{t_{i-1}}}\right) \right) \right. \\
&\quad \left. + e^{z_2(z_3 - z_1)} \left(n\left(z_3 \sqrt{t_i} + \frac{z_2}{\sqrt{t_i}}\right) \frac{1}{\sqrt{t_i}} - n\left(z_3 \sqrt{t_{i-1}} + \frac{z_2}{\sqrt{t_{i-1}}}\right) \frac{1}{\sqrt{t_{i-1}}} \right) \right] \frac{1}{\sigma x} \\
&- \frac{1}{2\nu} \left(\frac{z_1 - z_3}{z_3} \right) \left[e^{-z_2(z_3 + z_1)} (z_3 + z_1) \left(N\left(z_3 \sqrt{t_i} - \frac{z_2}{\sqrt{t_i}}\right) - N\left(z_3 \sqrt{t_{i-1}} - \frac{z_2}{\sqrt{t_{i-1}}}\right) \right) \right. \\
&\quad \left. + e^{-z_2(z_3 + z_1)} \left(n\left(z_3 \sqrt{t_i} - \frac{z_2}{\sqrt{t_i}}\right) \frac{1}{\sqrt{t_i}} - n\left(z_3 \sqrt{t_{i-1}} - \frac{z_2}{\sqrt{t_{i-1}}}\right) \frac{1}{\sqrt{t_{i-1}}} \right) \right] \frac{1}{\sigma x},
\end{aligned}$$

where

$$z_1 = \frac{(r - \delta - z + \phi \sigma^2 / 2)}{\sigma}, \quad z_2 = \frac{\ln(x/y)}{\sigma}, \quad z_3 = \sqrt{z_1^2 + 2\nu}.$$

To find the unknown coefficients A_{Mj} and a_{Mj} , we solve at each step j , for $j = 1, 2, \dots, M$, the system of equations (3.28) using a Newton-Raphson method.

The approximation procedure of European continuous-installment puts proceeds in the same way as for calls. Let \tilde{p}_t^M , for $t \in [0, T]$, denote the approximated initial premium of a European continuous-installment put corresponding to approximating the optimal stopping boundary G_s , for $s \in [t, T]$, by an M -piece exponential function $\{G_{Mj} e^{-g_{Mj}(T-t_j)}, j = 1, 2, \dots, M\}$. Then \tilde{p}_t^M is given by

$$\tilde{p}_t^M = \begin{cases} p(M, S_t, A, a, \phi, \nu, T - t), & \text{if } S_t < G_{MM} \\ 0, & \text{if } S_t \geq G_{MM} \end{cases}, \quad (3.29)$$

where

$$p(j, x, A, a, \phi, \nu, t_j - t) = p^{\text{BS}}(x; K, t_j - t) + q \sum_{i=1}^j I(t_{i-1}, t_i, x, G_{M, j-i+1}, -g_{M, j-i+1}, -1, r).$$

As for calls, applying the value-matching and high-contact conditions (3.11–3.12) at each time step t_j , we can determine the bases G_{Mj} and the exponents g_{Mj} , for $j = 1, 2, \dots, M$, of the M -piece exponential function.

4. Linear complementarity problem formulation

The European continuous-installment option pricing problem can be posed as a linear time-dependent complementarity problem of the form

$$\frac{\partial V_t}{\partial t} + \mu S_t \frac{\partial V_t}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - rV_t - q \leq 0, \quad (4.1)$$

$$V(S_t, t; q) \geq 0, \quad (4.2)$$

$$V(S_t, t; q) \left(\frac{\partial V_t}{\partial t} + \mu S_t \frac{\partial V_t}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_t}{\partial S^2} - rV_t - q \right) = 0, \quad (4.3)$$

with the final (or terminal) condition $V(S_T, T; q) = H(S_T)$. In contrast to the homogeneous Black-Scholes equation for the European vanilla option, the inequality (4.1), whose left-hand side is derived equating the returns on a hedge option portfolio and a bank account (after subtracting a constant amount $q dt$ to account for the installment premium), reflects

the asymmetric relationship between the long and short positions of the European continuous-installment option contract; only the holder has the right to terminate installment payments at any time before maturity. The inequality constraint (4.2) comes directly from the no-arbitrage principle that the option contract should not be worthless (if not, the holder just stop making installment payments and the option lapses).

In order to deal with the forward problem we define the time to maturity $\tau := T - t$, with $0 \leq t \leq T$, and denote by $V_\tau = V(S_t, T - \tau; q)$ the initial premium function in transformed time, so that the inhomogeneous PDE (2.3) becomes

$$\frac{\partial V_\tau}{\partial \tau} + q = \mu S_t \frac{\partial V_\tau}{\partial S} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V_\tau}{\partial S^2} - r V_\tau := \mathcal{L}_{\text{BS}} V_\tau, \quad (4.4)$$

where \mathcal{L}_{BS} represents the standard Black-Scholes spatial operator. Using the time variable τ the partial differential complementarity problem (4.1-4.3) can be rewritten as

$$\left(\frac{\partial}{\partial \tau} - \mathcal{L}_{\text{BS}} \right) V_\tau + q \geq 0, \quad (4.5)$$

$$V_\tau \geq 0, \quad (4.6)$$

$$V_\tau \left[\left(\frac{\partial}{\partial \tau} - \mathcal{L}_{\text{BS}} \right) V_\tau + q \right] = 0. \quad (4.7)$$

Within this setting the free boundary, along which the option holder must be indifferent between continuing and terminating installment payments, is dealt with abstractly so that the early stopping curve, which divides the domain \mathcal{D} into the continuation and stopping regions, can be computed once the initial premium has been found.

4.1. Finite difference approximation schemes

In order to discretize the partial differential complementarity problem (4.5-4.7) on a bounded computational domain $\Omega = \Omega_S \times \Omega_\tau$, the variable S_t , $\forall t \geq 0$, is truncated at a sufficiently large value \bar{S}_{\max} so as not to introduce huge errors in the approximate solution of the problem. Then the spatial domain $[\bar{S}_{\min}, \bar{S}_{\max}]$, with $0 \leq \bar{S}_{\min} < \bar{S}_{\max}$, and the time interval $[t, T]$ is divided into respectively m and n segments of equal lengths by the following discretizations

$$\Omega_S := \{S_i \in \mathbb{R}^+ \mid S_i = \bar{S}_{\min} + i\Delta S, i = 0, 1, \dots, m\},$$

$$\Omega_\tau := \{\tau_j \in \mathbb{R}^+ \mid \tau_j = t + j\Delta\tau, j = 0, 1, \dots, n\},$$

where $\Delta S := (\bar{S}_{\max} - \bar{S}_{\min})/m$ and $\Delta\tau := \tau/n$ are the space- and time- steps, respectively. Introducing the notation $V_i^j = V(S_i, \tau_j; q)$, we denote by $\mathbf{V}^j = [V_1^j, \dots, V_{m-1}^j]'$ the vector of unknowns at the interior grid points of Ω_S . Adopting the notation in [24], the finite difference approximation scheme to the Black-Scholes partial differential operator (4.4) can be described by

$$[\mathbf{I} - \theta\Delta\tau\mathbf{A}] \mathbf{V}^{j+1} + \Delta\tau \mathbf{q} = [\mathbf{I} + (1 - \theta)\Delta\tau\mathbf{A}] \mathbf{V}^j + \mathbf{g}, \quad (4.8)$$

where $0 \leq \theta \leq 1$ denotes the temporal weighting, $\mathbf{q} = [q, \dots, q]'$, \mathbf{A} is a tridiagonal matrix of order $m - 1$ defined by

$$\mathbf{A} := \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & \beta_2 & \gamma_2 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \alpha_{m-2} & \beta_{m-2} & \gamma_{m-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \alpha_{m-1} & \beta_{m-1} & \gamma_{m-1} \end{pmatrix},$$

with

$$\alpha_i = \frac{\sigma^2 S_i^2}{2\Delta S^2} - \frac{\mu S_i}{2\Delta S}, \quad \beta_i = \frac{-\sigma^2 S_i^2}{\Delta S^2 - r}, \quad \gamma_i = \frac{\sigma^2 S_i^2}{2\Delta S^2} + \frac{\mu S_i}{2\Delta S},$$

and

$$\mathbf{g} = [\alpha_1 \Delta \tau ((1 - \theta) V_0^j + \theta V_0^{j+1}), 0, \dots, 0, \gamma_{m-1} \Delta \tau ((1 - \theta) V_m^j + \theta V_m^{j+1})]'$$

incorporates the boundary conditions into the linear system (4.8). The finite difference approximation scheme is fully either implicitly or explicitly when the weighted average θ is equal to 1 or 0, respectively. The Crank-Nicolson finite difference (CNFD) method which is used to solve our pricing problem requires to set $\theta = 1/2$.

To determine the initial premium and early stopping curve implicitly we adopt the *explicit payout* technique, which requires to solve the linear system (4.8) with a direct method and then to truncate the solution only to positive (continuation) values in order to satisfy inequality (4.6).

5. Optimal stopping problem formulation

The valuation of European continuous-installment options can also be formulated as an optimal stopping problem. For $t \in [0, T]$, let $g = (g_\tau)_{\tau \in [t, T]}$ be the discounted net payoff process of a European continuous-installment option defined as

$$g(\tau) = \begin{cases} \mathbf{1}_{\{\tau=T\}}(e^{-r(T-t)}H(S_T)) - \frac{q}{r}(1 - e^{-r(\tau-t)}), & \text{if } r > 0 \\ \mathbf{1}_{\{\tau=T\}}(H(S_T)) - q(\tau - t), & \text{if } r = 0 \end{cases}, \quad (5.1)$$

where $H(S_T)$ is the plain vanilla payoff and q is a fixed non-negative installment per unit time. The first term on the right-hand side of equation (5.1) is the discounted payoff, received only if installment payments continue right up to the maturity date T , while the second term is the present value of installment premiums paid up to the time τ of termination. In the absence of arbitrage opportunities, the initial premium $V(S_t, t; q)$ to be paid at time $t \in [0, T]$, supplementary to a constant installment per unit time $q \geq 0$, is a solution of the following optimal stopping problem

$$V(S_t, t; q) = \sup_{\tau} \mathbb{E}_{t, S_t}^{\mathbb{Q}}(g_\tau), \quad (5.2)$$

with the supremum taken among all possible stopping times $\tau \in [t, T]$ and where $\mathbb{E}_{t, S_t}^{\mathbb{Q}}$ is the expectation operator at time t under the risk neutral probability measure \mathbb{Q} and conditional on the current asset value S_t . The optimal decision time $\tau^* \in [t, T]$ is defined as

$$\tau^* = T \wedge \tau_s^*,$$

where $\tau_s^* := \inf \{\tau_s \in [t, T] \mid V(S_{\tau_s}, \tau_s; q) = 0\}$ represents the optimal stopping time in the cancellation region. If $q = 0$, then $\tau^* = T$ (a.s.), in that it is not optimal to stop making installment payments before maturity.

5.1. Valuation with the Least-Squares Monte Carlo method

As for American continuous-installment options, the Least-Squares Monte Carlo (LSMC) method by [19] is modified in order to accommodate the pricing problem of European continuous-installment options.

Let us consider a discrete-time simple path for the underlying asset price $\{S_i = S(t_i), i = 0, 1, \dots, M\}$, with $t_i = t + i \Delta_t$ and where $\Delta_t := (T - t)/M$ is the time discretization. For a European vanilla option the price $v(S_t, t)$ is given by

$$v(S_t, t) = \mathbb{E}_{t, S_t}^{\mathbb{Q}}(e^{-r(T-t)}H(S_T)).$$

Since it does not allow for the possibility of early exercise, the price of the European vanilla option for each simulated time instant t_i corresponds to the expected continuation value. Therefore, at time step i and conditional on the current asset value S_i , the price (lump-sum premium) $v_i(S_i) = v(S_{t_i}, t_i)$ of the option contract is

$$v_i(S_i) = \mathbb{E}_i^{\mathbb{Q}}(e^{-r\Delta_t} v_{i+1}(S_{i+1}) \mid S_i),$$

where the function $v_i(\cdot)$ is defined recursively for $i = M - 1, M - 2, \dots, 0$. The value of $v_M(S_M)$ is simply $H(S_T)$, i.e., the plain vanilla payoff at maturity time $t_M = T$. Following [19], the conditional expectation of the continuation value $\mathbb{E}_{i, S_i}^{\mathbb{Q}}(\cdot)$ is approximated by a linear regression of the present value of $v_{i+1}(S_{i+1})$ at time step i on a set of polynomials of the current asset price S_i . To get observations for the regression, we have to replicate the sample path of the underlying asset price. The j th replication for the current asset price is denoted by S_i^j and correspondingly the j th replication

of the continuation value, which is the present value of $v_{i+1}^j(S_{i+1}^j)$, is denoted by y_i^j . Regressing on a second-order polynomial, the approximation of y_i^j is

$$y_i^j \cong \alpha_1 + \alpha_2 S_i^j + \alpha_3 (S_i^j)^2,$$

and the conditional expectation of the continuation value $\mathbb{E}_{i,S_i}^{\mathbb{Q}}(y_i^j)$ is given by $\hat{y}_i^j = \hat{\alpha}_1 + \hat{\alpha}_2 S_i^j + \hat{\alpha}_3 (S_i^j)^2$, where $\hat{\alpha}_k$, for $k = 1, 2, 3$, are the estimated regression coefficients.

When we consider a continuous-installment option with constant installment per unit time $q \geq 0$ and price (initial or up-front premium) denoted by $V_i(S_i) = V(S_i, t_i, q)$, the continuation value y_i^j becomes

$$e^{-r\Delta t} V_{i+1}^j(S_{i+1}^j) - \frac{q}{r}(1 - e^{-r\Delta t}),$$

and the same regression for the estimation of the conditional expectation \hat{y}_i^j can be used. The decision of early stopping at time step i and for sample path j is taken if

$$\hat{y}_i^j \leq 0, \quad j \in J_i^S,$$

where J_i^S is the set of paths that are out-of-the money at time step i . In particular, the set J_i^S constitute a partition of the set J of replicated paths. Therefore, the initial premium of the European continuous-installment option at time step i , conditional on the j th replication of the current price asset S_i^j , is given by

$$V_i^j(S_i^j) = \max \left\{ 0, \mathbb{E}_i^{\mathbb{Q}} \left(e^{-r\Delta t} V_{i+1}^j(S_{i+1}^j) \mid S_i^j \right) - \frac{q}{r}(1 - e^{-r\Delta t}) \right\}. \quad (5.3)$$

The computation of the option price is now achieved through the Algorithm 1, which provides a skeleton for the implementation of a computer code. Statements 5–8 consider the case where early stopping has to be checked. The set J^{S^*} corresponds to the paths where early stopping has taken place. Element j of array \mathcal{T} inform us about the time step where the early stopping decision has been taken for the j th path. In statement 10, the option price at time $t_0 = t$ for each path is saved in y_0^j , and, in statement 11, the average of these values is computed. The convergence of this method is analyzed in [11], where the choice of the order of the polynomial approximating the expectation operator $\mathbb{E}_i^{\mathbb{Q}}(\cdot)$ is discussed in conjunction with the number N of path replications and time steps M .

Algorithm 1

- 1: Generate $S \in \mathbb{R}^{N \times M}$
 - 2: Initialize $\mathcal{T}_j = M$ and $\mathcal{V}_j = H(S_M^j)$, for $j = 1, 2, \dots, N$
 - 3: **for** $i = (M - 1) \rightarrow 1$ **do**
 - 4: $y_i^j = e^{-r(\mathcal{T}_j - i)} \mathcal{V}_j - \frac{q}{r}(1 - e^{-r(\mathcal{T}_j - i)})$, for $j = 1, 2, \dots, N$
 - 5: Compute $J_i^S = \{j \mid y_i^j = 0\}$
 - 6: Estimate $\hat{y}_i^{j \in J_i^S} = \mathbb{E}_i^{\mathbb{Q}}(y_i^{j \in J_i^S} \mid S_i^{j \in J_i^S})$
 - 7: Compute $J_i^{S^*} = \{j \mid j \in J_i^S \wedge \hat{y}_i^{j \in J_i^S} \leq 0\}$
 - 8: Update $\mathcal{T}_j = i$ and $\mathcal{V}_j = 0$, for $j \in J_i^{S^*}$
 - 9: **end for**
 - 10: $y_0^j = e^{-r\mathcal{T}_j} \mathcal{V}_j - \frac{q}{r}(1 - e^{-r\mathcal{T}_j})$, for $j = 1, 2, \dots, N$
 - 11: $V_0(S_0) = \frac{1}{N} \sum_{j=1}^N y_0^j$
-

6. Numerical results and discussion

In this section we compare numerical results obtained with each of the three proposed methods and analyze the behavior of both the optimal stopping boundary and the initial premium with respect to the constant installment per unit time³.

³All algorithms have been implemented in Matlab 6.5

Table 1
Initial (up-front) premiums of European continuous-installment calls

σ	S_0	T	q	CNFD method	MEF method			LSMC method
					$M = 2$	$M = 6$	$M = 10$	(s.e.)
0.20	96	3/12	1	2.0671	2.0647	2.0668	2.0671	2.0671 (0.011)
			3	1.6607	1.6560	1.6602	1.6607	1.6603 (0.013)
			8	0.8035	0.7986	0.8035	0.8043	0.8038 (0.009)
		1	1	5.2323	5.2245	5.2308	5.2318	5.2321 (0.021)
			3	3.6495	3.6364	3.6473	3.6487	3.6492 (0.020)
			8	0.7154	0.7094	0.7152	0.7160	0.7150 (0.019)
	100	3/12	1	3.8343	3.8321	3.8339	3.8341	3.8340 (0.010)
			3	3.3845	3.3799	3.3838	3.3843	3.3844 (0.009)
			8	2.3378	2.3310	2.3371	2.3379	2.3380 (0.011)
		1	1	7.1999	7.1926	7.1984	7.1990	7.1989 (0.021)
			3	5.5107	5.4973	5.5083	5.5097	5.5103 (0.020)
			8	1.9836	1.9728	1.9823	1.9836	1.9835 (0.014)
104	3/12	1	6.2295	6.2278	6.2285	6.2292	6.2300 (0.016)	
		3	5.7558	5.7520	5.7550	5.7555	5.7556 (0.009)	
		8	4.6076	4.6011	4.6066	4.6074	4.6075 (0.008)	
	1	1	9.4775	9.4703	9.4761	9.4765	9.4773 (0.022)	
		3	7.7114	7.6985	7.7089	7.7103	7.7113 (0.014)	
		8	3.7678	3.7541	3.7658	3.7673	3.7676 (0.013)	
0.30	96	3/12	1	3.8976	3.8949	3.8971	3.8974	3.8975 (0.019)
			3	3.4668	3.4609	3.4659	3.4665	3.4666 (0.016)
			8	2.4794	2.4704	2.4784	2.4794	2.4795 (0.019)
		1	1	8.8982	8.8892	8.8963	8.8972	8.8977 (0.040)
			3	7.2435	7.2262	7.2404	7.2422	7.2432 (0.033)
			8	3.6802	3.6611	3.6774	3.6795	3.6793 (0.031)
	100	3/12	1	5.8017	5.7993	5.8012	5.8015	5.8012 (0.018)
			3	5.3476	5.3421	5.3466	5.3471	5.3473 (0.019)
			8	4.2685	4.2591	4.2671	4.2682	4.2681 (0.016)
		1	1	10.9795	10.9709	10.9777	10.9784	10.9791 (0.042)
			3	9.2661	9.2489	9.2628	9.2647	9.2660 (0.034)
			8	5.4322	5.4107	5.4288	5.4311	5.4318 (0.030)
	104	3/12	1	8.1256	8.1236	8.1251	8.1253	8.1254 (0.018)
			3	7.6559	7.6511	7.6549	7.6552	7.6556 (0.015)
			8	6.5161	6.5072	6.5146	6.5156	6.5159 (0.012)
		1	1	13.2649	13.2569	13.2631	13.2639	13.2646 (0.032)
			3	11.5044	11.4877	11.5010	11.5029	11.5039 (0.026)
			8	7.4560	7.4332	7.4521	7.4546	7.4557 (0.020)

This table reports and compares the initial (up-front) premiums at time $t = 0$ of European continuous-installment calls under the standard Black-Scholes assumptions using the three alternative numerical techniques. Initial premiums are calculated for two different levels of both volatility σ and time of maturity T , for three different values of constant installment per unit time q and for three different initial prices of underlying asset S_0 . Values chosen for common parameters are $K = 100$, $r = 0.05$ and $\delta = 0.04$. The CNFD method is implemented with 600 space steps between 0 and 200 for underlying asset price and 400 time steps per quarter of a year. The MEF method is tested for $M = 2$, $M = 6$ and $M = 10$. The LSMC method is based on 100 000 antithetic paths, a fourth-order Hermite polynomial for the regression and 80 time steps per quarter of a year.

The numerical results reported in Tables 1 and 2 show how the three alternative methods can efficiently and accurately approximate the initial (up-front) premiums of European continuous-installment calls and puts, respectively, under the standard assumptions of the Black-Scholes model. We consider option contracts written on an asset which pays a continuous dividend yield $\delta = 0.04$ with risk-free interest rate $r = 0.05$ and exercise price $K = 100$. The price at initial time $t = 0$ of the underlying asset, S_0 , is 96, 100 or 104, the time of maturity, T , is three months or one year, the instantaneous volatility of the percentage change in asset value, σ , is of 0.20 or 0.30 per annum and the constant

Table 2
Initial (up-front) premiums of European continuous-installment puts

σ	S_0	T	q	CNFD method	MEF method			LSMC method
					$M = 2$	$M = 6$	$M = 10$	(s.e.)
0.20	96	3/12	1	5.7588	5.7573	5.7585	5.7586	5.7587 (0.010)
			3	5.2827	5.2792	5.2820	5.2824	5.2825 (0.011)
			8	4.1259	4.1198	4.1250	4.1257	4.1255 (0.009)
		1	1	8.0621	8.0561	8.0608	8.0611	8.0619 (0.020)
			3	6.2717	6.2595	6.2693	6.2707	6.2716 (0.019)
			8	2.3033	2.2902	2.3015	2.3030	6.3029 (0.021)
	100	3/12	1	3.5860	3.5839	3.5856	3.5858	3.5857 (0.011)
			3	3.1332	3.1287	3.1325	3.1330	3.1331 (0.009)
			8	2.0779	2.0712	2.0772	2.0780	2.0781 (0.010)
		1	1	6.2342	6.2273	6.2327	6.2332	6.2341 (0.019)
			3	4.5178	4.5049	4.5155	4.5169	4.5176 (0.021)
			8	1.0209	1.0115	1.0200	1.0211	1.0213 (0.016)
104	3/12	1	2.0402	2.0378	2.0398	2.0401	2.0403 (0.013)	
		3	1.6271	1.6224	1.6265	1.6270	1.6270 (0.010)	
		8	0.7507	0.7457	0.7506	0.7512	0.7506 (0.008)	
	1	1	4.7140	4.7065	4.7125	4.7134	4.7137 (0.020)	
		3	3.0942	3.0813	3.0919	3.0933	3.0942 (0.016)	
		8	0.2452	0.2416	0.2456	0.2462	0.2454 (0.018)	
0.30	96	3/12	1	7.5962	7.5944	7.5958	7.5960	7.5961 (0.020)
			3	7.1212	7.1169	7.1203	7.1206	7.1215 (0.017)
			8	5.9627	5.9546	5.9613	5.9623	5.9630 (0.019)
		1	1	11.7355	11.7288	11.7340	11.7345	11.7351 (0.038)
			3	9.9169	9.9023	9.9138	9.9155	9.9166 (0.035)
			8	5.6666	5.6447	5.6628	5.6652	5.6653 (0.033)
	100	3/12	1	5.5521	5.5499	5.5517	5.5519	5.5519 (0.019)
			3	5.0911	5.0860	5.0902	5.0911	5.0907 (0.016)
			8	3.9880	3.9789	3.9866	3.9877	3.9876 (0.018)
		1	1	10.0017	9.9943	10.0000	10.0006	10.0013 (0.041)
			3	8.2209	8.2053	8.2177	8.2195	8.2204 (0.036)
			8	4.1638	4.1425	4.1604	4.1626	4.1631 (0.032)
	104	3/12	1	3.9272	3.9246	3.9267	3.9269	3.9270 (0.019)
			3	3.4858	3.4801	3.4848	3.4854	3.4851 (0.015)
			8	2.4619	2.4528	2.4607	2.4618	2.4622 (0.013)
		1	1	8.4705	8.4625	8.4687	8.4694	8.4710 (0.036)
			3	6.7343	6.7180	6.7311	6.7329	6.7338 (0.024)
			8	2.9100	2.8901	2.9069	2.9090	2.9098 (0.020)

This table reports and compares the initial (up-front) premiums at time $t = 0$ of European continuous-installment puts under the standard Black-Scholes assumptions using the three alternative numerical techniques. Initial premiums are calculated for two different levels of both volatility σ and time of maturity T , for three different values of constant installment per unit time q and for three different initial prices of underlying asset S_0 . Values chosen for common parameters are $K = 100$, $r = 0.05$ and $\delta = 0.04$. The CNFD method is implemented with 600 space steps between 0 and 200 for underlying asset price and 400 time steps per quarter of a year. The MEF method is tested for $M = 2$, $M = 6$ and $M = 10$. The LSMC method is based on 100 000 antithetic paths, a fourth-order Hermite polynomial for the regression and 80 time steps per quarter of a year.

installment per unit time, q , is 1, 3 or 8. For the Crank-Nicolson finite difference (CNFD) method we use 600 space steps between $\bar{S}_{\min} = 0$ and $\bar{S}_{\max} = 200$ for the underlying asset price and 400 time steps per quarter of a year. The Multipiece Exponential Function (MEF) method is tested for $M = 2$, $M = 6$ and $M = 10$. The Least-Squares Monte Carlo (LSMC) method is based on 100 000 antithetic paths, a fourth-order Hermite polynomial for the regression and 80 time steps per quarter of a year. Following [11, p. 18], these settings satisfy the conditions for convergence. To estimate the standard error (s.e.), we compute for each option contract a statistic with 50 initial premiums. The value reported in the table is the median of this statistic.

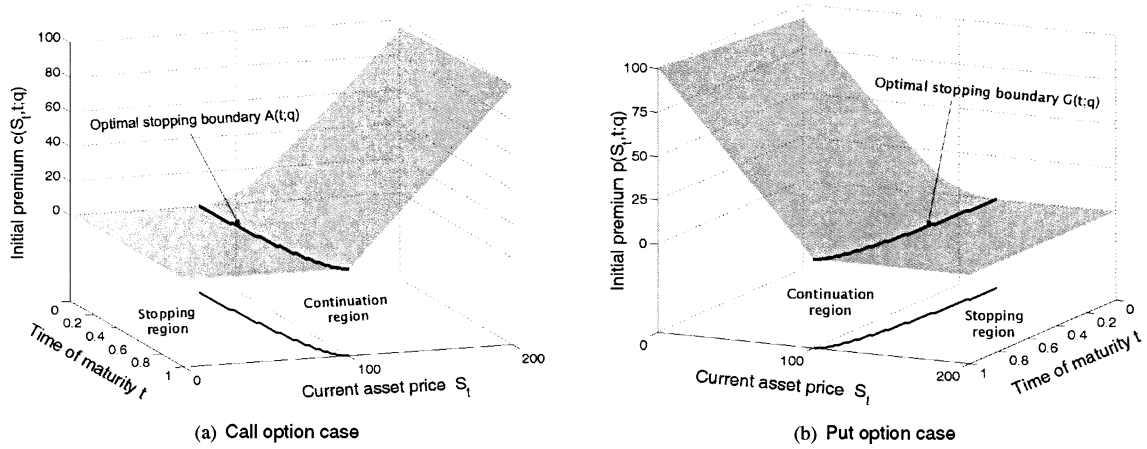


Figure 1

Initial premium surface, with optimal stopping boundary superimposed and projected on (S_t, t) plane.

The left and right graphs of this figure show the initial premium surfaces, with the optimal stopping boundary superimposed and projected on (S_t, t) plane, for European continuous-installment call and put, respectively. To determine the surfaces, initial premiums are calculated for different moneyness levels choosing the current asset price S_t from 0 to 200 and for different times of entering t from 0 to 1 year. Values chosen for common parameters are $K = 100$, $r = 0.05$, $\delta = 0.04$, $q = 3$, $\sigma = 0.20$ and $T = 1$. The CNFD method is implemented with 1200 space steps between 0 and 200 for underlying asset price and 800 time steps per quarter of a year.

Comparing the results obtained by the MEF method for $M = 10$ with those obtained by CNFD and LSMC methods, we see in Tables 1 and 2 that approximations of the option initial premium coincide from two to five digits. If the MEF method is tested for $M = 2$, the resulting approximations can have from one to three correct digits. In terms of computational efficiency, the CNFD method turns out to be the fastest with a computation time of less than 1 second to calculate initial premiums at all grid points for a three-months European continuous-installment option. The optimal stopping boundary can be derived implicitly from critical asset values on the space-time grid. The MEF method with $M = 10$ needs roughly 7 seconds to solve the pricing problem for the same option and provides the initial premium for a single value of S_t , as well as a pointwise approximation of the free boundary. If we consider $M = 2$, the computational time becomes comparable to that of the finite differences. Although the MEF method allows one to determine directly the two components in which the initial premium has been decomposed via integral representation method, an uncomfortable consequence may consist in the appropriate choice of initial values when one solving the non-linear system to find coefficients of the M -piece exponential function. The LSMC method spends approximately 11 seconds to calculate the initial premium. Since the result is of random nature we need to compute confidence intervals which imply repeated evaluations of the initial premium. An advantage of the LSMC method is that it can be easily extended to exotic payoffs and multifactor options.

The left and right panels of Figure 1 show the initial premium surfaces, with the optimal stopping boundary superimposed and projected on (S_t, t) plane, for European continuous-installment call and put options, respectively. To draw $c(S_t, t; q)$ - and $p(S_t, t; q)$ - surfaces, initial premiums are calculated for different moneyness and maturity levels, choosing respectively the current asset price S_t from 0 to 200 and the time of entering t from 0 to 1 year. Values chosen for common parameters are $K = 100$, $r = 0.05$, $\delta = 0.04$, $q = 3$, $\sigma = 0.20$ and $T = 1$. In order to ensure that $c(S_t, t; q)$ and $p(S_t, t; q)$ are smooth functions of S_t and t , the CNFD method is implemented with 1200 space steps between 0 and 200 for the underlying asset price and 800 time steps per quarter of a year. As we can see from Figures 1(a) and 1(b), as t approaches the maturity date T the initial premium approaches the hockey-stick payoff function $H(S_T)$, while the free boundary arising from the pricing problem reaches the limit value of the exercise price K . Onto the (S_t, t) plane the solid black line represents the free boundary projection, that is, the plot of the critical asset price against time t separating regions of optimally holding and stopping the option contract.

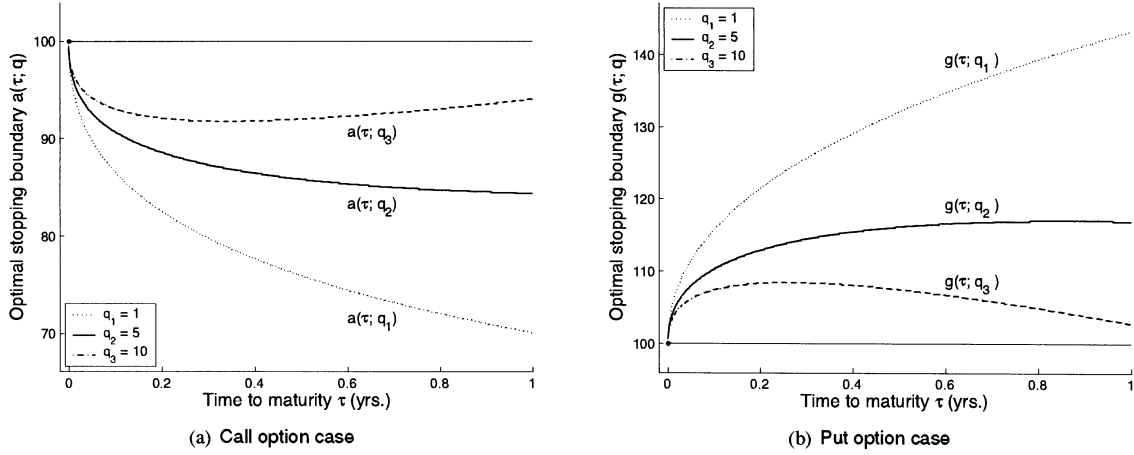


Figure 2

Optimal stopping boundaries for different levels of constant installment per unit time q .

The left and right graphs of this figure show the optimal stopping boundaries $a(\tau; q)$ and $g(\tau; q)$ of European continuous-installment call and put, respectively, as functions of time to maturity τ and for three different levels of constant installment per unit time q . To determine each curve, initial premiums are calculated for different moneyness levels choosing the current asset price S_t from 0 to 200 and for different times of entering t from 0 to 1 year. Values chosen for common parameters are $K = 100$, $r = 0.05$, $\delta = 0.04$, $\sigma = 0.20$ and $T = 1$. The CNFD method is implemented with 1200 space steps between 0 and 200 for underlying asset price and 800 time steps per quarter of a year.

The left and right panels of Figure 2 show the optimal stopping boundaries of European continuous-installment call and put options, respectively, as functions of time to maturity $\tau \in (0, 1]$ and for three different levels of constant installment per unit time q . To plot each curve, initial premium surfaces $c(S_t, t; q)$ and $p(S_t, t; q)$ are calculated for both the same bounded computational domain $\Omega = [0, 200] \times (0, 1]$ and set of parameters as in Figure 1, except for q which varies in $\{1, 5, 10\}$. Similarly, also the CNFD method is implemented with the same space- and time- discretizations. From Figures 2(a) and 2(b), we can prove that the optimal stopping boundaries $a(\tau; q)$ and $g(\tau; q)$ are, respectively, increasing and decreasing functions of q for all τ , and that their limiting behavior as $\tau \rightarrow 0^+$ is consistent with the theoretical result in Proposition 3.3. It is also clear from Figure 2 that the free boundaries are not monotone functions of time to maturity τ , namely $a(\tau; q)$ and $g(\tau; q)$ are, respectively, convex and concave functions of τ for all q . Finally, we observe that the continuation region becomes narrower as the installment per unit time q increases, since the early stopping curve approaches non-monotonically to the horizontal line at level K .

6.1. Convergence from discrete- to continuous-installment options

A further question of relevant interest is to investigate the limiting behavior of the initial premium function as the installment per unit time q increases. Intuitively, high values of q will reduce the range of underlying asset prices within which it is advantageous to continue making installment payments. Thus, as shown in Figure 2, the optimal stopping boundaries $a(\tau; q)$ and $g(\tau; q)$ tend to be closer to the exercise price K , for all time $\tau \in (0, T]$, and consequently the continuation region narrows.

[7] and [12] show that for European discrete-installment options with up-front premium p_0 at inception time $t_0 = 0$ and installment premiums p_1, p_2, \dots, p_{n-1} at subsequent decision dates $t_1, t_2, \dots, t_{n-1} \in (0, T)$, the uniform installment plan with $p_i \equiv p, \forall i = 0, 1, \dots, n-1$, converges to a finite upper bound $\hat{p} \in \mathbb{R}^+$ as the number n of installments increases. In fact, \hat{p} is the fair (or identical) premium for a European-style option contract with continuous payment plan, in which the holder pays a constant premium \hat{p} per unit time with the possibility to terminate payments at any time. This is equivalent to a portfolio of the corresponding vanilla option and an American put option on that contingent claim with time-varying strike price. As will be seen immediately below, the fair premium \hat{p} corresponds in our analysis to the upper bound of q for which the initial premium is null.

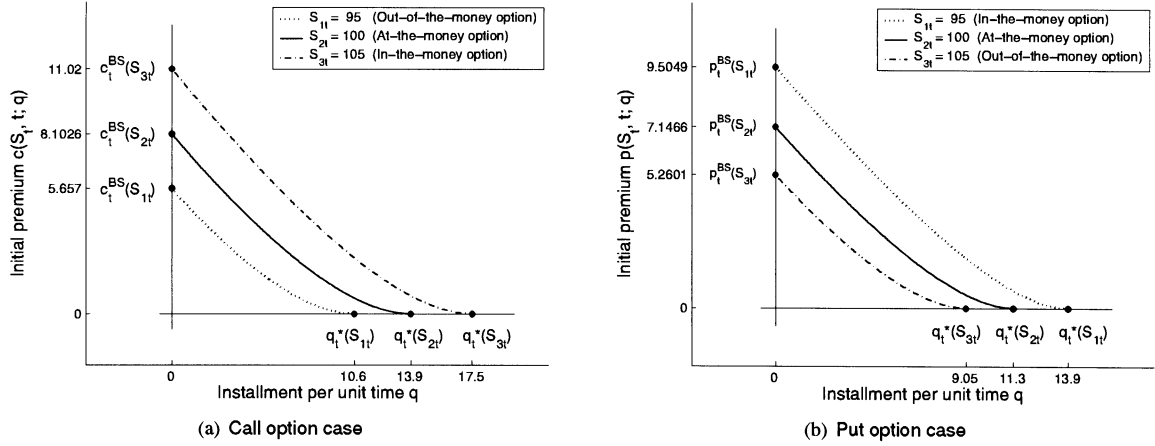


Figure 3

Initial premium as function of installment per unit time q .

The left and right graphs of this figure show the initial premiums $c(S_t, t; q)$ and $p(S_t, t; q)$ of European continuous-installment call and put, respectively, as functions of installment per unit time q and for three different levels of moneyness S_t . To determine each curve, initial premiums are calculated varying q from zero value, at which corresponds Black-Scholes call and put prices $c_t^{\text{BS}}(\cdot)$ and $p_t^{\text{BS}}(\cdot)$, to limit value $q_i^*(\cdot)$, choosing the current asset price $S_t \in \{95, 100, 105\}$ and setting time of entering $t = 0$. Values chosen for common parameters are $K = 100$, $r = 0.05$, $\delta = 0.04$, $\sigma = 0.20$ and $T = 1$. The CNFD method is implemented with 1200 space steps between 0 and 200 for underlying asset price and 800 time steps per quarter of a year.

Let us define the limit value of q as the *minimal installment* per unit time $q_i^* = q^*(S_t, t)$ such that the initial premium $V(S_t, t; q)$ of a European continuous-installment option is equal to zero, that is,

$$q_i^* = \inf \{q \in \mathbb{R}^+ \mid V(S_t, t; q) = 0\}, \quad (6.1)$$

where (S_t, t) is a fixed point of the continuation region. From equation (5.2), we note that the initial premium $V(S_t, t; q)$ can never become negative, since the option holder has the right to terminate installment payments at any time before maturity. In order to obtain a unique no-arbitrage value for q_i^* one has to solve the optimization problem (6.1), for which the existence and uniqueness of the solution are proved by [12]. While for American continuous-installment options can be demonstrated analytically that q_i^* is unbounded for all $t \geq 0$, namely only for $q \rightarrow \infty$ the continuation region vanishes and the initial premium function coincides with the option payoff, for European continuous-installment options the upper bound of q for each fixed point of the continuation region is obtained by a suitable numerical procedure that find the value q_i^* satisfying the minimality condition (6.1).

To determine the limit value q_i^* of a European continuous-installment call and put option, we draw the initial premium functions $c(S_t, t; q)$ and $p(S_t, t; q)$ against the installment per unit time q in the left and right panels of Figure 3, respectively. For each curve, initial premiums are calculated varying q from zero value, at which correspond the Black-Scholes call and put prices $c_t^{\text{BS}}(\cdot)$ and $p_t^{\text{BS}}(\cdot)$, to limit value $q_i^*(\cdot)$, choosing the current asset price $S_t \in \{95, 100, 105\}$ and setting the time of entering $t = 0$. Values chosen for common parameters are $K = 100$, $r = 0.05$, $\delta = 0.04$, $\sigma = 0.20$ and $T = 1$. The CNFD method with 1200 space steps between 0 and 200 for the underlying asset price and 800 time steps per quarter of a year is used to obtain these numerical results. We observe from Figure 3 that the initial premium is a strictly decreasing function of q , for both option types and all three chosen values of S_t , and that the limit value q_i^* increases with the absolute moneyness of the option.

Table 3 shows the fair premiums per unit time of European continuous-installment call and put options. To compute each fair premium we use the CNFD method with 2400 space steps between 0 and 200 for the underlying asset price and 1600 time steps per quarter of a year as an iterative procedure to solve the root finding problem (6.1). It follows that a high degree of accuracy and then long computational times are required. In order to examine the effects

Table 3

Fair premiums per unit time of European continuous-installment calls and puts

σ	r	δ	T	Continuous-installment calls			Continuous-installment puts		
				$S_0 = 96$	$S_0 = 100$	$S_0 = 104$	$S_0 = 96$	$S_0 = 100$	$S_0 = 104$
0.20	0.00	0.00	3/12	17.1606	26,4313	37.4614	34.5348	24.9527	17.0405
			6/12	14.0219	18,9716	24.5473	22.0179	17.4812	13.5629
			9/12	12.2230	15,6561	19.4294	17.0779	14.1599	11.5768
	0.03	0.02	3/12	17,5830	26,9469	38,0545	33,7637	24,2920	16,5079
			6/12	14,4395	19,4533	25,0831	21,2498	16,7997	12,9711
			9/12	12,6134	16,1117	19,9250	16,3049	13,4600	10,9535
	0.05	0.03	3/12	17,9867	27,5091	38,7122	33,0495	23,6866	16,0240
			6/12	14,9100	19,9986	25,6991	20,5541	16,1851	12,4414
			9/12	13,1054	16,6481	20,5157	15,6241	12,8424	10,4029
0.30	0.00	0.00	3/12	30,2145	40,2022	51,3750	45,8537	36,8495	29,0751
			6/12	23,6796	29,0409	34,8167	29,8943	25,6832	21,8933
			9/12	20,3353	24,0773	28,0364	23,3956	20,7149	18,2561
	0.03	0.02	3/12	30,6280	40,6759	51,8986	45,0800	36,1898	28,4701
			6/12	24,0656	29,4617	35,2664	29,1039	24,9551	21,2289
			9/12	20,6899	24,4551	28,4489	22,5882	19,9589	17,5541
	0.05	0.03	3/12	31,0940	41,2160	52,4890	44,3805	35,5631	27,8440
			6/12	24,5319	29,9777	35,8271	28,4121	24,3202	20,6429
			9/12	21,1468	24,9507	28,9798	21,8979	19,3118	16,9471

This table reports and compares the fair premiums at time of entering $t = 0$ of European continuous-installment calls and puts. Fair premiums are calculated for two levels of volatility σ (0.20 and 0.30), three times of maturity T (3, 6 and 9 months), three values of both interest rate r (0, 0.03 and 0.05) and dividend yield δ (0, 0.02 and 0.03) and three initial asset prices S_0 (96, 100 and 104). Value chosen for common parameter is $K = 100$. The CNFD method is implemented with 2400 space steps between 0 and 200 for underlying asset price and 1600 time steps per quarter of a year.

on the limit value q_t^* of model parameters, we consider two levels of volatility σ (0.20 and 0.30), three times of maturity T (3, 6 and 9 months), three values of both interest rate r (0, 0.03 and 0.05) and dividend yield δ (0, 0.02 and 0.03) and three initial asset prices S_0 (96, 100 and 104). As we can see from Table 3, the limit value $q_t^* = q^*(S_t, t)$ is a monotone function of both arguments, that is, q_t^* increases as the moneyness level increases, and decreases as the time to maturity increases. Specifically, the fair premium of in-the-money options with short-maturity (three months) is greater than that of at-the-money and out-of-the money options with medium- and long-maturities (six and nine months). Finally, a comparison of fair premiums across several values of parameters, shows that as the drift term $\mu = (r - \delta)$ increases the limit value q_t^* increases and decreases for installment call and put options, respectively, while higher levels for the volatility σ imply greater values of q_t^* for both option types. Since are well-known in the literature the relationships between the vanilla option prices (that form, along with the expected present value of payment stream, the installment option prices) and the model parameters, the revealed effects of r , δ and σ on the limit value q_t^* can be largely expected.

7. Conclusions

In this paper, we presented three alternative characterizations of the European continuous-installment options pricing problem, where the aim is to determine the initial (or up-front) premium given a constant installment payment plan.

Within the standard Black-Scholes model, we first formulated the pricing problem as a free boundary problem for the inhomogeneous Black-Scholes equation, and then using the integral representation method we obtained integral expressions for both the initial premium and the optimal stopping boundary, for which some analytical properties were derived. The Multipiece Exponential Function (MEF) method allowed an approximation in closed-form to the pricing formulas. Second, we adopted the linear complementarity formulation of the pricing problem for determining the initial premium and early stopping curve implicitly by solving the discretized problem on a bounded computational

domain using finite difference approximation schemes. Specifically, the Crank-Nicolson finite difference (CNFD) method in conjunction with the explicit payout technique was performed. Finally, we showed how the optimal stopping problem formulation for European continuous-installment options can be expressed, and then implemented by the Least-Squares Monte Carlo (LSMC) method.

Numerical tests are presented to show the effectiveness of the proposed numerical methods and to validate the theoretical analysis expressed in the three alternative characterizations of the pricing problem. Furthermore, numerical results permitted to analyze the convergence from discrete- to continuous-installment options, and to confirm that a European-style option contract with continuous payment plan is equivalent to a portfolio of the corresponding vanilla option and an American put option on that contingent claim with time-varying strike price.

The focus of this paper is on European continuous-installment options written on assets without dividends or with constant continuous dividend yield. However, by presenting a mathematically and computationally meaningful way to solve the early stopping problem of European options, this study enhances applications of the contingent-claims approach to investment problems in general. For example, investments involving a constant payment stream that can be stopped at any time may be readily analyzed using the framework developed in this paper.

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