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DEFAULT DEPENDENCE STRUCTURE EFFECTS ON THE VALUATION OF GOVERNMENT GUARANTEES

Carlo D. Mottura - Luca Passalacqua

Default dependence structure effects on the valuation of government guarantees

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Abstract

The paper analyses the problem of evaluating a guarantee contract against default risk in which the guarantor party is defaultable and the default risks of the guarantor and of the borrower are correlated. This problem has several relevant applications within the present sovereign risk crisis. We have investigated the effects of the dependence structure between defaults events within a framework defined by the classical no-arbitrage market approach, considering intensity models driven by Cox processes for the term structure of survival probabilities and copula models to derive the joint distribution of default times. We compare numerical results on the probability of the guarantee being paid, for different values of the default intensities, using the Gaussian and the Marshall-Olkin copulas, finding relevant differencies and counter-intuitive dependence on the correlation parameter.

Keywords: government guarantees, default risk, correlation, Marshall-Olkin.

JEL Classification: C160, G130, G280.

1 The problem

We consider the problem of evaluating a guarantee contract where both the guarantor and the borrower are subject to default risk and these risks are correlated.

This problem has several relevant applications within the present sovereign risk crisis, *e.g.* in the case of guarantees issued by euro area member states to back liabilities of the European Stability Mechanism (ESM [2]) or of government guarantees to back banks' liabilities (for Italy see *e.g* [6]).

At present, ESM provides financial assistance to euro area member states in financial difficulties by raising funds principally issuing bonds with maturities up to 30 years. These bonds are backed by a total subscribed capital of 700 billion euros. Of this amount, 80 billion is the paid-in capital and 620 billion is the committed callable capital. The committed callable capital, *i.e.* the European government guarantees, is shared among the euro area member states on a pro-rata basis, based on the ECB contribution key, and is free of charge for ESM. From the government debt point of view these guarantees are public contingent liabilities on the ESM credit risk. However, according to Eurostat rules, the ESM liabilities will not increase the government debt of the shareholder countries.
As far as government guarantees to Italian banks' liabilities are concerned, art. 8 of [6] ("*Misure per la stabilità del sistema creditizio*" – measures for improving the stability of the banking system) allows the Italian Ministry of Economy to provide State guarantees on bank liabilities, under the supervision of the Italian Central Bank. Similarly to the ESM case, this type of guarantee, from the guarantor point of view, is a public contingent liability on banks credit risk. Data on euro member states contingent liabilities are available in [3].

As a matter of fact, at least from a legal point of view, this type of guarantee is quite peculiar, since *(a)* both parties are exposed to counterparty default risk and the risk of the guarantor could be higher than the one of the borrower, and *(b)* because of the structure of dependence that might exists between the two risks.
For example, in the case of Italy – that is the third shareholder of the ESM, after Germany and France, with a contribution of 125 billion euros – we could consider a stand-alone guarantee provided to a liability of the ESM, Italy's rating being lower then that of the ESM.
Another case of interest is that of a sovereign bond bought by a bank when bank's liabilities are guaranteed by the sovereign issuer, so that the higher is the State's default risk the higher will be the bank's one, and viceversa.
The financial literature [17] suggests that "the market value of a sovereign guarantee is not only a positive function of the weakness of the borrower but also a positive function of the creditworthiness of the sovereign. Thus, to avoid competitive distortions, the strength of the sovereign should be taken into account in the pricing of government-provided guarantees".
However, in addition to what suggested in [17], in a more general context, the value of the contract should also depend on the creditworthiness dependence structure. Therefore, we have addressed the question of the effects of different default dependence structures on the valuation of a defaultable and correlated government guarantee.

2 Modelling framework

The problem of the valuation of a defaultable government guarantee is analogous to the valuation of credit default swaps (CDS) subject to counterparty risk, that is widely discussed in the literature with several approaches ranging to the use of copula functions to contagion models (see *e.g* [16], [10]). In the words of the European Central Bank "the increased correlation in the CDS market between reference entities and sellers of CDS protections lessens the effectiveness of clean transfer of risk and amplifies the effect of these interconnectedness. This risk, called *wrong-way risk*, occurs when the creditwhorthiness or credit quality of a CDS reference entity is correlated with the CDS counterpart's ability or willingness to pay ... Stress testing should be used to identify any *wrong-way risk* in existing portfolios, with risk mitigance and/or the adjustment of capital employed to reflect any existing *wrong-way risk* ... This *wrong-way* risk could, for example, apply to affiliates within the same corporate group, but could in priciple also apply to wholly separate legal entities which are exposed to similar economic or external risks" [1].

It is clear that the exposure of the guarantor to default lowers the price of the guarantee, and that, as in the case of a CDS, the valuation of the contract depends on the sequence of default times. However, depending on the payoff of the contract, the effects may be very different. The difference between the price of the contract obtained in case of default-free guarantor and defaultable guarantor is an optional component of the contract, known as counterparty-risk credit valuation adjustment (CVA).

In the following, we consider the payoff of a defaultable and correlated government guarantee, paid by the guarantor party (party # "1") to a guaranteed party at the time of default of the borrower (party # "2"), if this occurs before the maturity of the contract:

$$\begin{aligned}\Pi(\tau_2 \wedge T) &= N\big[1 - R(\tau_2 \wedge T)\big]\, \mathbb{1}_{\{\tau_1 > \tau_2\}}\, \mathbb{1}_{\{\tau_2 \le T\}} \\ &= \begin{cases} N\big[1 - R(\tau_2 \wedge T)\big], & \text{if} \quad \tau_2 \le T \quad \text{and} \quad \tau_1 > \tau_2, \\ 0 \qquad\qquad\qquad\quad, & \text{if} \quad \tau_2 > T \quad \text{or} \quad \tau_1 \le \tau_2, \end{cases}\end{aligned} \tag{1}$$

where $(\tau_2 \wedge T) = \min\{\tau_2, T\}$, T is the maturity of the guarantee, N is the guaranteed liability, $R(t)$ is the (possibly) stochastic recovery rate at time t in case of default determined on the basis of the book asset-liability unbalance of the borrower, $\tau_1 \ge 0$ and $\tau_2 \ge 0$ are, respectively, the times of default of the guarantor and of the borrower, that are supposed to be correlated. The payoff in (1) corresponds to zero recovery rate of the guarantor, and, in this sense, can be further generalised.

In a standard setting defined within a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$ and under the classical no-arbitrage market approach, the value of the guarantee at the valuation time t is given by:

$$V(t) = \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau_2 \wedge T} r(u)du}\, \Pi(\tau_2 \wedge T) \Big| \mathcal{G}_t \right], \tag{2}$$

where $r(t)$ is the (possibly) stochastic risk-free interest rate intensity (spot rate) prevailing on the market, $(\mathcal{G}_t)_{t \ge 0}$ is the relevant filtration representing the flow of information of the whole market and $\mathbb{Q}$ is the risk-neutral measure.

As discussed in the next section, we further simplify the above theoretical framework by using an intensity model with constant intensities for the term structure of survival probabilities of both parties and considering two standard copulas, namely the Gaussian copula – a standard reference for the corporate market – and the Marshall-Olkin copula, which is particularly suited to describe *systemic* market shocks.

We consider this setting to be driven by reasonable requirements, giving at the same time conceptual insightfulness and a degree of flexibility which allows a minimal empirical consistency. The comparison of the two copulas will allow to measure the effects of the structure of dependence between default events on the valuation of the guarantee, that is the primary goal of this work.

In this framework, we focus our attention on the risk-neutral probability of event A:

$$\mathbf{P}[A] = \mathbf{P}\big[(\tau_2 \leq T) \cap (\tau_1 > \tau_2)\big], \tag{3}$$

i.e. the probability of the guarantee being paid, which is the key ingredient for the valuation of the contract. In fact, the valuation of $\mathbf{P}[A]$ requires the knowledge of the joint distribution of default times, that is easily derived in our framework by using Monte Carlo simulations. Similarly, the value of the guarantee $V(t)$ can be easily obtained by discounting the payoff obtained in each simulation, according to the risk free term structure prevailing on the market.

Finally, we point out that the payoff of the contract can be generalized to the case of m multi-garantor parties, $G^{(1)}, \ldots, G^{(m)}$ and a single borrower D:

$$\Pi_k(\tau_2 \wedge T) = \min\Big\{ \frac{\omega_k \mathbb{1}_{\{\tau_k > \tau_D\}}}{\sum_{k=1}^{m} \omega_k \mathbb{1}_{\{\tau_k > (\tau_2 \wedge T)\}}} N\Big[1 - R(\tau_2 \wedge T)\Big] \mathbb{1}_{\{\tau_D \leq T\}}, \bar{C}_k(t_0) \Big\}, \tag{4}$$

where we have supposed that the guarantors share *pro quota* the guarantee up to a maximum amount $\bar{C}_k(t_0)$ contractually fixed at the issue time t_0 ($k = 1, \ldots, m$), although other mechanisms could also be considered, as *e.g.* with the *pari passu* clause. In any case, whenever $m > 1$, it is necessary to describe the default dependence structure among the guarantors. A multidimensional copula setting is, again, well suited for such a task.

2.1 Joint default times distribution

A canonical way to derive the joint distribution of default times is to use a copula model. Copulas of dimension d [15] are distribution functions on $[0,1]^d$ with standard uniform marginal distributions. Their importance is due to the famous Sklar theorem that states that for any given distribution function F with continuous margins $F_1, \ldots F_d$ there exists a unique copula $C : [0,1]^d \mapsto [0,1]$ such that:

$$F(x_1, \ldots, x_d) = C\big(F_1(x_1), \ldots, F_d(x_d)\big), \quad \forall (x_1, \ldots, x_d) \in \mathbb{R}^d. \tag{5}$$

Survival distribution functions $\bar{F}$ can similarly be espressed in terms of copulas and survival function marginals $\bar{F}_1, \ldots, \bar{F}_d$. In fact, by definition:

$$\begin{aligned} \bar{F}(x_1, \ldots, x_d) &= \mathbf{P}\Big[X_1 > x_1, \ldots, X_d > x_d] \\ &= \mathbf{P}\Big[1 - F(X_1) \leq \bar{F}(x_1), \ldots, 1 - F(X_d) \leq \bar{F}(x_d)\Big] \\ &= \widehat{C}\Big(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)\Big) \end{aligned} \tag{6}$$

where $\widehat{C}$ is named the *survival* copula. The relationship between a copula C and its associated survival copula $\widehat{C}$ is obvious in the univariate case:

$$\widehat{C}(u_1) = 1 - C(u_1) = 1 - u_1, \tag{7}$$

while the general case is discussed in [4].

Notice that since the survival copula is itself a copula, financial models can be formulated equivalently by specifying the copula or by specifying the survival copula. Using copulas allows to build the joint distribution of default times by:

1. sampling random vectors $(u_1, \ldots, u_d) \in [0, 1]^d$ with a given survival copula C as joint distribution function;

2. obtain the default times $(\tau_1, \ldots, \tau_2) \in \mathbb{R}^d_+$ by:

$$\tau_i = F_i^{-1}(u_i), \tag{8}$$

   where $F_i(x)$ is the term structure of survival probabilities for the i-th obligor ($i = 1, \ldots, d$). Classical exogenous choices of the survival functions $F_i(x)$ are *e.g.* the exponential, Weibull, log-normal, log-logit and Gompertz distributions. In intensity models driven by Cox processes, as pioneered by Lando [7], survival probabilities at time t are expressed as a function of the (possibly) stochastic default intensity $\lambda(x)$ as:

$$F_i(x) = \mathbb{1}_{\tau > t} \mathbb{E}\Big[\exp\Big(-\int_t^x \lambda(u)du\Big) \mid \mathcal{F}_t\Big], \tag{9}$$

   where $\mathcal{F}_t \subseteq \mathcal{G}_t$ is the filtration containing "*default-free*" information, so that the time of default is a totally inacessible (*i.e.* "unpredictable") stopping time. Clearly, in the limiting case of a constant intensity λ this setting reduces to the choice of an exponential survival function. More complex choices, *e.g.* where $\lambda(t)$ follows a Lévy process, have been considered in the literature.

From a technical point of view, Monte Carlo implementation of steps 1. and 2. is easily performed in very short computing time.

The advantage of using copulas is that they allow to disantangle the dependence structure from the marginals. Financially speaking, this means that the term structure of default probabilities can be inferred from single name credit derivatives, typically credit default swaps or defaultable bonds, while the parameters of the copula are inferred in a second stage from multi-name credit derivatives, typically collateralised debt obligations (CDO's) or credit indices.

The choice of the models to be used in the two stages is driven by usual requirements such as parsimony, tractability, empirical consistency and conceptual insightfulness. For sovereign obligors it is common practice to use intensity models to extract (risk neutral) implicit default probabilities from the term structure of CDS spreads, while – due to the scarcity of market data – there is no clear consensus on the choice of the copula.

2.1.1 The Gaussian copula

The multi-name credit derivatives market standard reference is the Gaussian copula formulated by Li [9]. When the marginal distributions are univariate normal the Li model is exactly J.P. Morgan CreditMetrics model expressed in the copula framework. In turn, CreditMetrics can be considered one possible extension of the (univariate) Merton model [13].

The Gaussian copula function C_G is defined as:

$$C_G(u_1, \dots, u_d) = \Phi_d\Big(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d); \Sigma\Big), \tag{10}$$

where $\Phi(x)$ (resp. $\Phi_d(x_1, \dots, x_d)$) is the univariate (resp. multivariate) distribution function of a standard normal random variable and Σ is a correlation matrix, for which it is customary to assume homogeneous pairwise correlations ρ. In the bivariate case this copula reduces to:

$$C_G(u_1, u_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \exp\Big(-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\Big) dx_1 dx_2. \tag{11}$$

The model can be further formulated as a so-called *conditionally independent* default model, according to which there exist a single normally distributed market variable $M \sim N(0,1)$ and d normally distributed idiosyncratic variables $\varepsilon_i \sim N(0,1)$, such that:

$$\begin{array}{lcll} U_i & = & \Phi(Z_i), & \\ Z_i & = & \sqrt{\rho_i}\, M + \sqrt{1-\rho_i}\, \varepsilon_i, & \\ \mathbf{Cov}[M, \varepsilon_i] & = & 0, & i = 1, \dots, d, \\ \mathbf{Cov}[\varepsilon_i, \varepsilon_j] & = & 0, & i = 1, \dots, d,\ j = 1, \dots, d,\ i \neq j. \end{array} \tag{12}$$

Default correlations are induced by the common dependence of the latent variables Z_i (corresponding to the standardised log-return value of the firms in the Merton model) to the market variable M, so that:

$$\mathbf{Cor}[Z_i, Z_j] = \sqrt{\rho_i \rho_j}, \quad i \neq j. \tag{13}$$

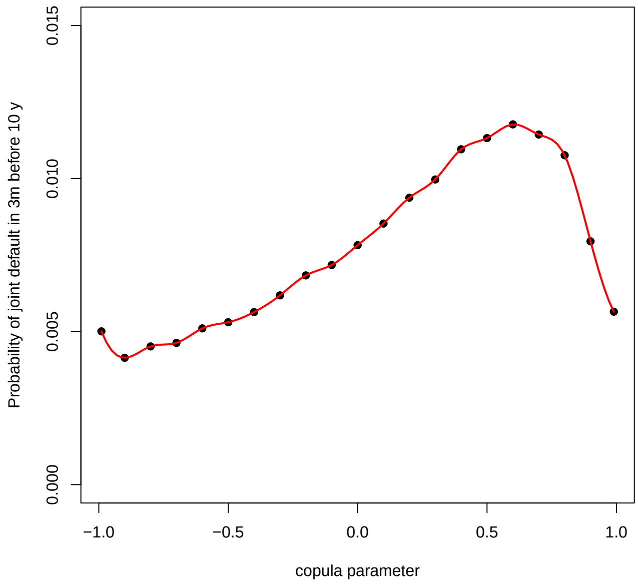
Notice that the parameters ρ_i $(i = 1, \dots, d)$ measure the correlation between the latent variables, and that this correlation is different from the correlation between default times or from the correlation between default indicators; *i.e.* in general:

$$\mathbf{Cor}[Z_i, Z_j] \neq \mathbf{Cor}[\tau_i, \tau_j] \neq \mathbf{Cor}[\mathbb{1}_{\{\tau_i \leq T\}}, \mathbb{1}_{\{\tau_j \leq T\}}], \quad i \neq j. \tag{14}$$

Unfortunately, for the Gaussian copula there are no closed form expressions for both the correlation between default times and the correlation between default indicators.

It appears quite relevant the fact that the concept of "*correlated default times*" is not mathematically precise. It embeds the idea that the second (later) default time is triggered by the same process that caused the first (earlier) default time, but it does not clarify if the time lag between two events should satisfy some properties. Events with a fixed but large time delay $\Delta t = \max\{\tau_1, \tau_2\} -$

Figure 1: Probability of having default times before T = 10 years and within a time lag of $\Delta t = 3$ months, obtained with a Gaussian copula with exponential marginals, as a function of the copula parameter ρ. The default intensities are respectively $\lambda_1 = 0.20$ years^{-1} and $\lambda_2 = 0.02$ years^{-1}.



$\min\{\tau_1, \tau_2\}$, *e.g.* $\Delta t = 100$ years, are clearly fully correlated, but do not correspond to scenarios relevant for valuation of credit derivatives and their corresponding risks. In other words, the definition of *correlation* to be used must be suited with respect to the *payoff* of the contract considered.

For what we are concerned, we shall point out, that for the Gaussian copula with independent marginal default intensities the are two relevant issues. Firstly, the probability of joint defaults in a given short period is not monotone in the correlation parameter ρ. For example, figure 1 shows the probability of having both obligors in default within $\Delta t = 3$ months:

$$\mathbf{P}\Big[\{\tau_1 \leq T\} \cap \{\tau_2 \leq T\} \cap \{|\tau_1 - \tau_2| \leq \Delta t\}\Big], \tag{15}$$

as a function of ρ in the bivariate case, for exponential survival probabilities with $\lambda_1 = 0.20$ years^{-1}, $\lambda_2 = 0.02$ years^{-1} and $T = 10$ years.

Secondly, the Gaussian copula is unable to produce joint defaults close in time ($\Delta t \to 0$) when the default intensities are different.

Morini [14] has discussed this issue in the framework of the credit crisis pointing out that a sequence of defaults close in time would most presumably trigger a *flight to quality* mechanism, if not a "*panic*" reaction, from the market whilst the same number of defaults, more spaced in time, would probably not.

Notice that we have refered to *independent marginal defaul intensities*. In fact, it is possible to formulate models in which the default of one obligors modifies

– with a typical *contagion* mechanism (see *e.g.* [8]) – the default intensity of another obligor, or models in which there exist a *systemic* source of risk able to trigger *simultaneous* defaults. An example belonging to the second class of models is the Marshall-Olkin model.

2.1.2 The Marshall-Olkin copula

In the bivariate Marshall-Olkin model [11] there are two obligors subject to default and three independent Poisson processes N_t^1, N_t^2 and N_t^{12} with intensities $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_{12} \geq 0$.

The default of obligor i ($i = 1, 2$) can arrive either at the first jump time $X_i = \inf\{t \geq 0 : N_t^i > 0\}$ of the Poisson process N_t^i, or at the first jump time $X_{12} = \inf\{t \geq 0 : N_t^{12} > 0\}$ of the Poisson process N_t^{12}, which triggers the simultaneous default of both obligors. Therefore, letting τ_i ($i = 1, 2$) the default time of the i-th obligor, then

$$\tau_i = \min\{X_i, X_{12}\} \quad i = 1, 2.$$

The joint distribution of default times (τ_1, τ_2) is called the Marshall-Olkin bivariate exponential distribution with parameters $(\lambda_1, \lambda_2, \lambda_{12})$. The corresponding survival copula is:

$$C_{MO}(u_1, u_2) = u_1 u_2 \min\{u_1^{-\alpha_1}, u_2^{-\alpha_2}\}, \tag{16}$$

where $\alpha_i = \lambda_{12}/(\lambda_i + \lambda_{12})$ ($i = 1, 2$). Marginal distributions are exponential with intensities $\Lambda_i = \lambda_i + \lambda_{12}$ ($i = 1, 2$). It is then clear that the default intensities receive a contribution from a common "systemic" factor.

In this case the correlation between default times and default indicators are available in closed form:

$$\mathbf{Cor}[\tau_1, \tau_2] = \frac{\Lambda_1 + \Lambda_2 - \Lambda_{12}}{\Lambda_{12}} = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}, \tag{17}$$

$$\mathbf{Cor}[\mathbb{1}_{\{\tau_1 < t\}}, \mathbb{1}_{\{\tau_2 < t\}}] = \sqrt{\frac{e^{-(\Lambda_1 + \Lambda_2)t}}{(1 - e^{-\Lambda_1 t})(1 - e^{-\Lambda_2 t})}} \left(e^{\lambda_{12} t} - 1\right), \tag{18}$$

where $\Lambda_{12} = \lambda_1 + \lambda_2 + \lambda_{12}$. Moreover, it easy to show that the probability of simultaneous defaults is equal to the correlation in default times:

$$\mathbf{P}[\tau_1 = \tau_2] = \mathbf{Cor}[\tau_1, \tau_2]. \tag{19}$$

From eq. (17) it is clear that independence is obtained for $\lambda_{12} = 0$ and maximal dependence in the limit $(\lambda_1 + \lambda_2) \to 0$. However, since $\lambda_{12} \leq \min\{\Lambda_1, \Lambda_2\}$, the maximal attainable correlation between defaul times is:

$$\bar{\rho} = \max_{\Lambda_1, \Lambda_2, \lambda_{12}} \mathbf{Cor}[\tau_1, \tau_2] = \frac{\min\{\Lambda_1, \Lambda_2\}}{\max\{\Lambda_1, \Lambda_2\}}. \tag{20}$$

Referring to the previous numerical example in this case the probability of simultaneous defaults can be as big as:

$$\bar{\rho} = \frac{0.02}{0.20} = 0.1, \tag{21}$$

depending on the estimate of λ_{12}. For a general discussion on attainable correlations, see *e.g.* [12].

Multi-dimensional extensions of the Marshall Olkin copula are described *e.g.* in [10]. In this case the shocks can be either idiosyincratic, global or limited to any given sub-sample of obligors, thus mimicking the segmentation of the market.

3 Numerical results

We have provided numerical results on the probability of the guarantee being paid $\mathbf{P}[A]$, for different values of the default intensities, using the Gaussian (G) and the Marshall-Olkin (MO) copulas.

Inspired by current values of implied risk-neutral default probabilities in sovereign CDS, we have considered as a reference level of the borrower default intensity $\lambda_2 = 0.02$ years^{-1}, that roughly corresponds to a AA rating. For the guarantor party we have considered four different reference levels, namely $\lambda_1 = 0.01, 0.02, 0.06$, and 0.2 years^{-1}, which roughly correspond to the credit standings of Germany, France, Italy and Greece. Notice that the case $(\lambda_1, \lambda_2) = (0.01, 0.02)$ is the only case in which the creditworthiness of the guarantor party is higher than that of the borrower.

For each set of default intensities and dependence structure (copula) we have computed the probability of the guarantee being paid by using 500,000 Monte Carlo simulations. Results are reported in table 1 for the two copulas as a function of the attainable correlation ρ between default times. Figure 2 illustrates the eight cases (4 couples of intensities times 2 copulas).

For the Gaussian copula we notice that the value of the probability *(a)* decreases – as expected – with the weakness of the guarantor, and *(b)* contrary to naive expectations, it is dependent to the correlation parameter ρ in different ways: it increases with ρ for $\lambda_1 < \lambda_2$ and decreases in the opposite case. This facet, that also appears in the valuation of CDS counterparty risk (see *e.g.* [5]), is due to the fact that, at full correlation ($\rho = 1$) the party that defaults first is always the one with the higher default intensity. Thus, if it is the guarantor to default first, the value of the guarantee should vanish, whereas it reaches its maximum value in the opposite case.

Differently, for the Marshall-Olkin copula the value of the probability is always decreasing with increasing values of ρ. Moreover, the values obtained with the Marshall-Olkin copula are always significantly smaller than those obtained with the Gaussian one, except when $\rho = 0$ when, by definition, they should coincide. For example, for $(\lambda_1, \lambda_2) = (0.01, 0.02)$ years^{-1} and $\rho = 0.5$, the value of the probability obtained with the Marshall-Olkin copula is about 1.7 smaller than that obtained with the Gaussian copula. These features are easily understood recalling that the Marshall-Olkin copula has a discontinuity in the joint distribution of default times at $\tau_1 = \tau_2$, that increases in size with increasing values of the correlation parameter.

Tables 2 and 3 and figures 3 and 4 are analogous of table 1 and figure 2, respectively, but for levels of $\lambda_2 = 0.04$ and 0.06 years^{-1}. The previous comments on the dependence on ρ apply also in these cases. In addition, as expected, for both copulas, the value of the probability increases with the weakness of the borrower, for a fixed level of guarantor creditworthiness.

Finally, for all cases, notice that the value of the probability is significant even when the creditworthiness of the guarantor is lower than that of the borrower.

4 Conclusions

We have considered the problem of evaluating a government guarantee contract where both the guarantor party and the borrower are subject to default risk and these risks are correlated.
We have addresses this problem using a well established framework making use of the classical no-arbitrage market approach, an intensity model driven by Cox processes for the term structure of survival probabilities and a copula model to describe the joint distribution of default times. We have assumed this setting to be driven by reasonable requirements in terms of parsimony, tractability, conceptual insightfulness and minimal empirical consistency. However, analysing two different copulas, namely the Gaussian copula – a standard reference for the corporate market – and the Marshall-Olkin copula, we have shown evidence that for the Gaussian copula with independent marginal default intensities there are two relevant effects on the valuation of the guarantee contract. Firstly, the probability of the guarantee being paid can increase with the correlation between the default events when the default intensity of the guarantor is lower than that of the borrower, while it decreases in the opposite case. This is contrary to (our) naive expectations since the guarantee is expected to be worthless – or at least greatly reduced – if the guarantor has defaulted. Differently, with the Marshall-Olkin copula the value of the probability is always descreasing with increasing correlation in default, in agreement with expectations. Moreover, the values obtained with the Marshall-Olkin copula are always significantly smaller than those obtained with the Gaussian one.

Therefore, it appears that the default dependence structure is of paramount importance in the valuation of defaultable and correlated governement guarantees and that the observation of only the credit standings of the parties is not sufficient to perform the valuation of this type of contracts, but that proper care should be given in modelling the dependence structure of defaults.

Table 1: Value of the probability of the guarantee being paid for different values of the default intensity of the guarantor, λ_1, and as a function of the attainable linear correlation betweeen default times, ρ, for a given value of the default intensity of the borrower $\lambda_2 = 0.02$ years^{-1}. Default intensities are expressed in years^{-1}, probabilities are in %.

	$\lambda_2 = 0.02$ $P(\tau_2 \leq T) = 45.12$							
	$\lambda_1 = 0.01$		$\lambda_1 = 0.02$		$\lambda_1 = 0.06$		$\lambda_1 = 0.20$	
ρ	G	MO	G	MO	G	MO	G	MO
0.00	39.57	39.61	35.07	34.92	22.73	22.73	9.01	9.07
0.05	39.38	37.47	34.41	32.46	22.12	19.14	8.47	4.55
0.10	38.97	35.43	34.01	29.83	21.29	15.53	7.94	0.00
0.15	38.70	33.53	33.53	27.49	20.57	12.11	7.31	–
0.20	38.40	31.64	33.10	25.31	19.84	8.68	6.74	–
0.25	38.20	30.04	32.59	23.04	18.91	5.32	6.18	–
0.30	37.97	28.18	32.12	21.11	18.29	2.10	5.60	–
0.35	37.72	26.84	31.62	19.06	17.49	–	5.04	–
0.40	37.72	25.36	31.32	17.23	16.53	–	4.40	–
0.45	37.49	23.94	30.99	15.51	15.59	–	3.84	–
0.50	37.40	22.57	30.40	13.74	14.59	–	3.31	–
0.55	37.27	–	29.95	12.13	13.54	–	2.79	–
0.60	37.21	–	29.38	10.52	12.39	–	2.25	–
0.65	37.17	–	28.91	9.00	11.16	–	1.79	–
0.70	37.38	–	28.45	7.56	9.73	–	1.35	–
0.75	37.47	–	27.89	6.18	8.27	–	0.92	–
0.80	37.86	–	27.35	4.84	6.53	–	0.55	–
0.85	38.64	–	26.19	3.57	4.56	–	0.26	–
0.90	40.06	–	26.94	2.38	2.48	–	0.08	–
0.95	42.50	–	22.60	1.19	0.70	–	0.01	–
1.00	45.21	–	0.00	0.00	0.00	–	0.00	–

Figure 2: Probability of guarantee being paid as a function of linear correlation between default times, for different values of the default intensities ($\lambda_1, \lambda_2 = 0.02$) and the two copulas. The maturity of the contract is $T = 30$ years. The full symbols stands for the Marshall-Olkin (MO) copula while the empty symbols represent the results obtained with the Gaussian (G) copula. The values of the four sets of default intensities used for each copula are reported in years^{-1}. Each point is obtained with 500,000 Monte Carlo simulations.

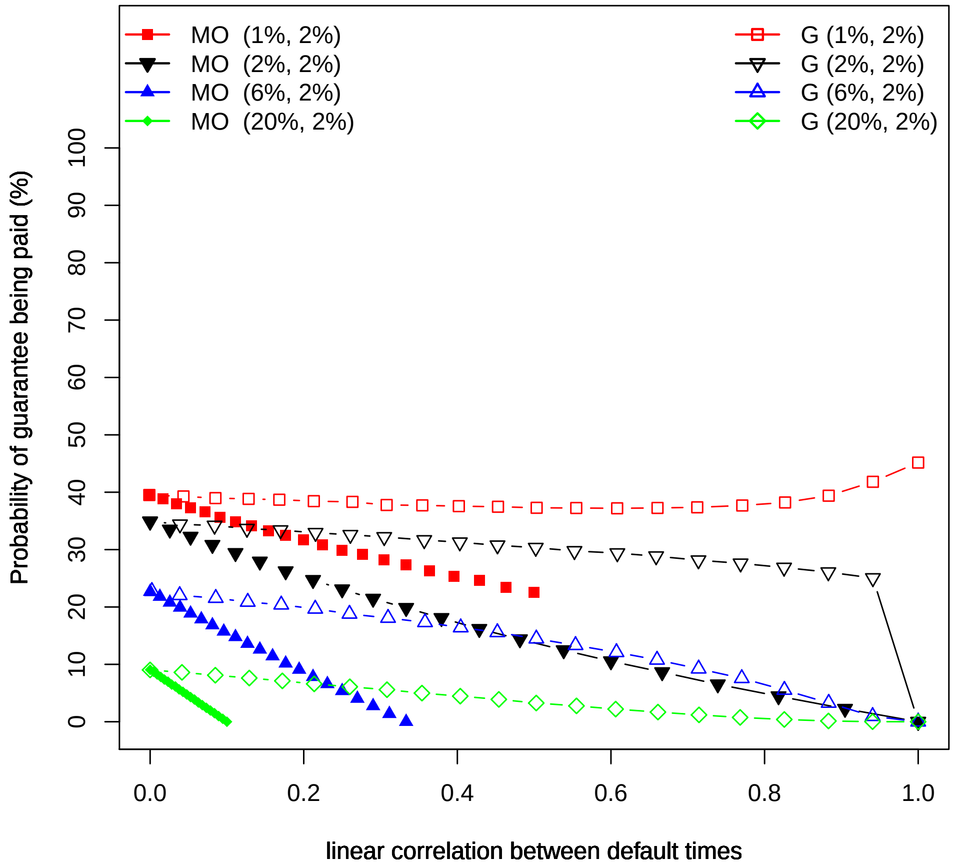


Table 2: Value of the probability of the guarantee being paid for different values of the default intensity of the guarantor, λ_1, and as a function of the attainable linear correlation betweeen default times, ρ, for a given value of the default intensity of the borrower $\lambda_2 = 0.04$ years^{-1}. Default intensities are expressed in years^{-1}, probabilities are in %.

	$\lambda_2 = 0.04$ $P(\tau_2 \leq T) = 69.88$							
	$\lambda_1 = 0.01$		$\lambda_1 = 0.02$		$\lambda_1 = 0.06$		$\lambda_1 = 0.20$	
ρ	G	MO	G	MO	G	MO	G	MO
0.00	62.09	62.18	55.63	55.65	37.90	37.91	16.58	16.62
0.05	62.07	60.12	55.48	53.21	37.55	34.88	15.97	12.45
0.10	62.04	58.04	55.22	51.09	36.84	31.81	15.23	8.28
0.15	61.92	56.06	55.10	48.71	36.48	28.76	14.49	4.18
0.20	61.97	54.25	55.06	46.66	35.84	25.74	13.68	0.00
0.25	62.08	52.52	54.94	44.54	35.30	22.73	12.97	–
0.30	62.00	–	54.86	42.51	34.86	19.79	12.06	–
0.35	62.24	–	54.85	40.41	34.01	16.98	11.30	–
0.40	62.44	–	54.93	38.63	33.34	14.09	10.36	–
0.45	62.66	–	55.05	36.72	32.60	11.44	9.43	–
0.50	62.99	–	55.24	–	31.85	8.74	8.48	–
0.55	63.36	–	55.54	–	31.06	5.99	7.50	–
0.60	63.82	–	55.67	–	30.09	3.40	6.48	–
0.65	64.25	–	56.03	–	28.83	0.85	5.45	–
0.70	65.04	–	56.71	–	27.67	–	4.35	–
0.75	65.78	–	57.56	–	26.14	–	3.31	–
0.80	66.72	–	58.70	–	24.15	–	2.30	–
0.85	67.80	–	60.25	–	21.57	–	1.37	–
0.90	68.81	–	62.80	–	17.99	–	0.59	–
0.95	69.65	–	66.43	–	11.41	–	0.08	–
1.00	69.89	–	69.92	–	0.00	–	0.00	–

Figure 3: Probability of guarantee being paid as a function of linear correlation between default times, for different values of the default intensities ($\lambda_1, \lambda_2 = 0.04$) and the two copulas. The maturity of the contract is $T = 30$ years. The full symbols stands for the Marshall-Olkin (MO) copula while the empty symbols represent the results obtained with the Gaussian (G) copula. The values of the four sets of default intensities used for each copula are reported in years^{-1}. Each point is obtained with 500,000 Monte Carlo simulations.

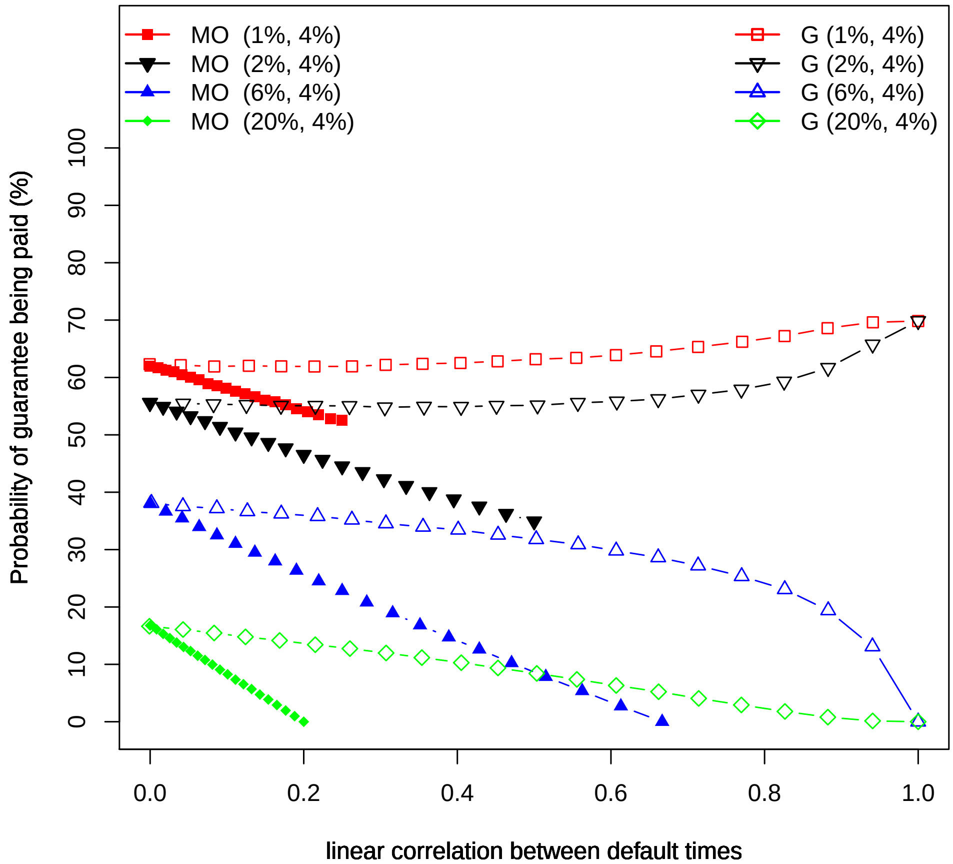
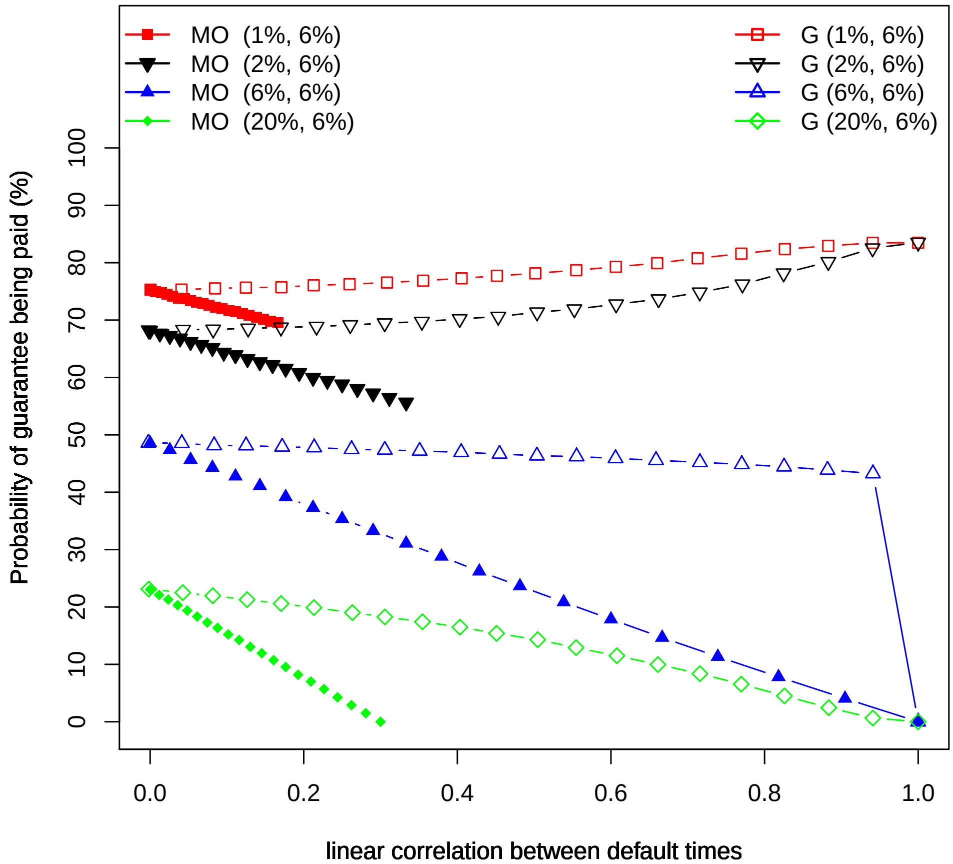


Table 3: Value of the probability of the guarantee being paid for different values of the default intensity of the guarantor, λ_1, and as a function of the attainable linear correlation betweeen default times, ρ, for a given value of the default intensity of the borrower $\lambda_2 = 0.06$ years^{-1}. Default intensities are expressed in years^{-1}, probabilities are in %.

	$\lambda_2 = 0.06$ $P(\tau_2 \leq T) = 83.47$							
	$\lambda_1 = 0.01$		$\lambda_1 = 0.02$		$\lambda_1 = 0.06$		$\lambda_1 = 0.20$	
ρ	G	MO	G	MO	G	MO	G	MO
0.00	75.27	75.26	68.17	68.15	48.57	48.54	23.11	23.07
0.05	75.31	73.65	68.35	66.24	48.39	46.04	22.27	19.25
0.10	75.50	71.82	68.40	64.34	48.24	43.32	21.61	15.27
0.15	75.62	70.15	68.63	62.41	47.90	40.57	20.92	11.57
0.20	75.89	–	68.94	60.69	47.92	37.95	20.04	7.66
0.25	76.27	–	69.12	58.75	47.66	35.39	19.29	3.79
0.30	76.45	–	69.37	56.82	47.30	32.84	18.35	0.00
0.35	76.81	–	69.72	–	47.18	30.16	17.43	–
0.40	77.29	–	70.29	–	46.91	27.68	16.44	–
0.45	77.60	–	70.67	–	46.75	25.24	15.40	–
0.50	78.10	–	71.23	–	46.31	22.72	14.27	–
0.55	78.68	–	71.78	–	46.05	20.25	13.13	–
0.60	79.28	–	72.59	–	45.95	17.93	11.75	–
0.65	79.90	–	73.51	–	45.57	15.49	10.46	–
0.70	80.44	–	74.49	–	45.27	13.15	8.90	–
0.75	81.20	–	75.80	–	44.88	10.89	7.16	–
0.80	81.98	–	77.31	–	44.89	8.60	5.48	–
0.85	82.57	–	79.02	–	43.53	6.39	3.69	–
0.90	83.09	–	81.02	–	45.50	4.27	1.80	–
0.95	83.43	–	82.75	–	39.46	2.14	0.46	–
1.00	83.42	–	83.50	–	0.00	0.00	0.00	–

Figure 4: Probability of guarantee being paid as a function of linear correlation between default times, for different values of the default intensities $(\lambda_1, \lambda_2 = 0.06)$ and the two copulas. The maturity of the contract is $T = 30$ years. The full symbols stands for the Marshall-Olkin (MO) copula while the empty symbols represent the results obtained with the Gaussian (G) copula. The values of the four sets of default intensities used for each copula are reported in years^{-1}. Each point is obtained with 500,000 Monte Carlo simulations.



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