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A NOTE ON PREDICTION MARKETS

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A NOTE ON PREDICTION MARKETS

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A NOTE ON PREDICTION MARKETS

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Abstract

In a prediction market, individuals can sequentially place bets on the outcome of a future event. This leaves a trail of personal probabilities for the event, each being conditional on the current individual's private background knowledge and on the previously announced probabilities of other individuals, which give partial information about their private knowledge. By means of theory and examples, we revisit some results in this area. In particular, we consider the case of two individuals, who start with the same overall probability distribution but different private information, and then take turns in updating their probabilities. We note convergence of the announced probabilities to a limiting value, which may or may not be the same as that based on pooling their private information.

JEL Classification: C53, D8, G14

Some key words: consensus; expert opinion; information aggregation; probability forecast; sequential prediction

1 Introduction

This paper revisits some of the results appearing in economic literature from a statistical point of view.

A prediction market – also known as a predictive market, an information market, a decision market, or a virtual market – is a venue where actors trade predictions on uncertain future events and can also allow participants to stake bets on the likelihood of various events occurring. These events include, for example, an election result, a terrorist attack, a natural disaster, commodity prices, quarterly sales or even sporting outcomes. Prediction markets also offer trade in possible future outcomes on securities markets, in which case participants who use it are buying something like a futures contract. The Iowa Electronic Markets (<http://tippie.uiowa.edu/iem/>) of the University of Iowa Henry B. Tippie College of Business is one of the main prediction markets in operation.

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Also companies like Google have setup their own internal prediction markets. Prediction markets sometimes operate on an open market like the stock market, or in a closed market akin to a betting pool. A prediction market translates the wisdom of crowds into predictive probabilities.

For example, suppose that in a prediction market one can bet whether A occurs (before time t), where actors buy and sell contracts among each other. Let a contract pay 1 if event A occurs and 0 otherwise. Say, the market price for the contract is 0.58. Offers to buy and sell are fixed at 0.57 and at 0.59, respectively. Now, you can either pay 0.59 instantly, or post an offer to pay 0.58 and see if any actor is willing to sell at that price. Now, the current market price, 0.58, is the consensus probability.

Prediction markets have been studied by Aldous (2013); Arrow et al. (2008); Hanson (2003); Chen et al. (2010); Hanson et al. (2006); Wolfers and Zitzewitz (2008); Strähl and Ziegel (2015); Ehm et al. (2016), among others.

2 Basic setup

We shall focus on the opinions of a specific individual, “You”, and how these opinions change in the light of accumulating experience. Taking a fully Bayesian position, we suppose that Your opinions are expressed as a joint probability distribution, \Pr , over all relevant variables. Other individuals may have their own probabilities for various events, but for You these are treated simply as potential data. In the sequel, all probabilities are computed under Your distribution \Pr .

We shall interpret the term “expert” in the sense of DeGroot (1988); Dawid et al. (1995). That is, an individual E is an expert (for You) if E started with exactly the same joint probability distribution \Pr over all relevant variables as You, and has observed everything that you have observed, and possibly more. If You now learn (just) the probability Π that E assigns to some event A , your updated probability for A will be Π . That is, You will agree with the expert.

In the context of a prediction market, experts E_1, E_2, \dots , sequentially give their probability predictions Π_1, Π_2, \dots , for an uncertain event A . E_i is the expert that makes the forecast at time i . We allow for the case that it could be the same expert giving his forecast at different times. At time i expert E_i has access to all previous forecasts Π_1, \dots, Π_{i-1} , and additional private information H_i . E_i will typically not have access to the private information sets H_1, \dots, H_{i-1} that the previous experts used in formulating their forecasts, but only to the actual forecasts made. However, in some markets there is an option for forecasters to leave comments, which could give additional partial information K_i (which might be empty) about H_i . We assume that each forecaster is aware of all such past comments. Thus $\Pi_i = \Pr(A | T_i)$, where $T_i := (K_1, \Pi_1, \dots, K_{i-1}, \Pi_{i-1}, H_i)$ is the total information available to E_i .

The full public information available just after time i is $S_i := (K_1, \Pi_1, K_2, \Pi_2, \dots, K_i, \Pi_i)$.

Note that S_i and T_i both contain all the information made public up to time $i - 1$. They differ only in the information they contain for time i : here T_i specifies the totality, H_i , of expert E_i 's information, both public, K_i , and private, whereas S_i specifies only E_i 's public information, K_i , and her announced probability forecast, Π_i , for A at time i . The information sets (T_i) are not in general increasing with i , since H_i is included in T_i but need not be in T_{i+1} . The information sets (S_i) are however increasing. The following Lemma and Corollary show that, for You, for the purposes of predicting A both information sets T_i and S_i are equivalent, and Your associated prediction is just the most recently announced probability forecast.

Lemma 1 $\Pr(A | S_i) = \Pr(A | T_i) = \Pi_i$.

Proof. Since $T_i \supset S_i \ni \Pi_i$,

$$\begin{aligned} \Pr(A | S_i) &= E\{\Pr(A | T_i) | S_i\} \\ &= E(\Pi_i | S_i) \\ &= \Pi_i \\ &= \Pr(A | T_i). \end{aligned}$$

□

Corollary 1 *If You observe the full public information S_i , and have no further private information, Your conditional probability for A is just the last announced forecast Π_i .*

3 Convergence

From Lemma 1 and the fact that the information sequence (S_i) is increasing, we have:

Corollary 2 *The sequence (Π_i) is a martingale with respect to (S_i).*

Then by Corollary 2 and the martingale convergence theorem, we now have:

Corollary 3 *As $i \rightarrow \infty$, Π_i tends to a limiting value Π_∞ .*

The variable Π_∞ is random in the sense that it depends on the initially unknown (to You) information sequence $S_\infty := \lim S_i$ that will materialise, but will be a fixed value for any such sequence.

A perhaps surprising implication of Corollary 3 is that, eventually, introduction of new experts will not appreciably change the probability You assign to A — whatever new private information they may bring will be asymptotically negligible compared with the accumulated public information. We may term Π_∞ the *consensus probability* of A , and the information S_∞ on which it is based the *consensus information set*.

The information S_∞ is *common knowledge* for all experts in the sense of Aumann (1976). For details See Geanakoplos (1992a,b); Nielsen (1984); McKelvey and Page (1986).

It might be considered that the limiting value Π_∞ has succeeded in integrating all the private knowledge of the infinite sequence of experts. As we shall see below this is sometimes, but not always, the case.

4 Two experts

As a special case, suppose we have a finite set of experts, E_1, \dots, E_N , and we take $E_{N+1} = E_1$ (so $H_{N+1} = H_1$), $E_{N+2} = E_2$, etc.. Thus we repeatedly cycle through the experts. Continuing for many such cycles, eventually we will get convergence, to some Π_∞ — at which point each expert will not be changing her opinion based on the total sequence of publicly announced forecasts, even though she still has access to additional private information.

At convergence, it will thus make no difference to Expert E_i to incorporate (again) her private information H_i . Consequently we have:

Proposition 4 *For each i , $A \perp\!\!\!\perp H_i \mid S_\infty$.*

Dutta and Polemarchakis (2014) give a simple example with two experts, that shows that the order in which the experts play can matter. In their example they show that when one of the experts starts playing they reach a complete consensus, whereas changing the order in which they play they reach a limited consensus. Dutta and Polemarchakis (2014) also show that if an expert has additional information this can lead to a weaker consensus. They call this “obfuscation”.

In the sequel we consider in detail the case $N = 2$ of two experts, who alternate $E_1, E_2, E_1, E_2, \dots$ in updating and announcing their forecasts. Geanakoplos and Polemarchakis (1982) have studied this in the case that there is no side-information, and each expert E_i 's set of possible private information has finite cardinality, K_i say. They show that exact consensus is reached in at most $K_1 + K_2$ rounds.

4.1 Vacuous consensus

We start with some examples where the experts learn nothing from each other's forecasts—although they would learn more if they were able to communicate and pool their private data.

Example 1 Parity check

This example is essentially the same as that described by Geanakoplos and Polemarchakis (1982, p. 198).

Let X_1, X_2 be independent fair coin tosses. Expert E_i observes only X_i ($i = 1, 2$). Let A be the event $X_1 = X_2$. This has prior probability 0.5.

On observing his private information X_1 , whatever value it may take, E_1 's probability of A is unchanged, at 0.5. His announcement of that value is therefore totally uninformative about the value of X_1 . Consequently E_2 can only condition on her private information about X_2 —which similarly has no effect. The sequence of forecasts will thus be 0.5, 0.5, 0.5, \dots . Convergence is immediate, but to a vacuous state.

However, if the experts could pool their data, they would learn the value of A with certainty. \square

Example 2 Bivariate normal

With this example, we generalise from predicting an uncertain event to predicting an uncertain quantity.

Suppose that E_1 observes X_1 , and E_2 observes X_2 , where (X_1, X_2) have a bivariate normal distribution with means $E(X_i) = 0$, variances $\text{var}(X_i) = 1$, and unknown correlation coefficient ρ —which is what they have to forecast. Let ρ have a prior distribution Π_0 . Since X_1 is totally uninformative about ρ , E_1 's first forecast is again Π_0 , and so is itself uninformative. Again, E_2 has learned nothing relevant to ρ , and so outputs forecast Π_0 ; and so on, leading to immediate convergence to a vacuous state. However the pooled data (X_1, X_2) is informative about ρ (though does not determine ρ with certainty). \square

In the above examples, each expert's private information was marginally independent of the event or variable, generically Y say, being forecast, with the immediate result that the consensus forecast was vacuous, the same as the prior forecast. Conversely, suppose the consensus is vacuous. That is to say,

$$Y \perp\!\!\!\perp S_\infty. \tag{1}$$

But from Proposition 4 (trivially generalised) we have

$$Y \perp\!\!\!\perp H_i \mid S_\infty. \tag{2}$$

Combining (1) and (2), we obtain $Y \perp\!\!\!\perp (H_i, S_\infty)$ whence, in particular,

$$Y \perp\!\!\!\perp H_i.$$

Hence the consensus will be vacuous if and only if each expert's private information is, marginally, totally uninformative. The argument extends trivially to any finite number of experts.

4.2 Complete consensus

We use the term *complete consensus* to refer to the case that the consensus forecast will be the same as the forecast based on the totality of the private information available to all the individual forecasters. A simple situation where this will occur is when Π_i is a one-to-one function of H_i , so that, by announcing Π_i , expert E_i fully reveals her private information.

Example 3 Overlapping Bernoulli trials

Let θ be a random variable with a distribution over $[0, 1]$ having full support. Given θ , let $Y_0 \sim B(n_0, \theta)$, $Y_1 \sim B(n_1, \theta)$, $Y_2 \sim B(n_2, \theta)$, and $A \sim B(1, \theta)$, all independently.

Suppose E_1 observes $X_1 = Y_0 + Y_1$, and E_2 observes $X_2 = Y_0 + Y_2$. At the first stage, E_1 computes and announces $\Pi_1 = \Pr(A | X_1)$ —which is a one-to-one function of X_1 . For example, under a uniform prior distribution for θ , $\Pi_1 = (X_1 + 1)/(n_0 + n_1 + 2)$. Then at stage 2, E_2 will have learned X_1 , and also has private information X_2 . Thus $\Pi_2 = \Pr(A | X_1, X_2)$, the correct forecast given the complete private information of E_1 and E_2 . Further cycles will not change this probability, which will be the consensus. \square

Example 4 Linear prediction

Consider variables $\mathbf{X} = (X_1, \dots, X_k)$, $\mathbf{Z} = (Z_1, \dots, Z_h)$ and (scalar) Y , all being jointly normally distributed with non-singular dispersion matrix. Expert 1 observes $H_1 = \mathbf{X}$, Expert 2 observes $H_2 = \mathbf{Z}$, and they have to forecast Y . Each time an expert announces her predictive distribution for Y , she is making known the value of her predictive mean of Y , which will be some linear combination of the predictor variables (\mathbf{X}, \mathbf{Z}) . So generically we would expect convergence of the forecasts, after at most $\min\{k, h\}$ rounds, to the full forecast based on the pooled information (\mathbf{X}, \mathbf{Z}) .

In order to investigate this we have made use of the 93CARS dataset (Lock, 1993), containing information on new cars for the 1993 model year. There are $n = 82$ complete cases with information on 26 variables, including price, mpg ratings, engine size, body size, and other features. We took $\mathbf{X} = (X_1, \dots, X_{11})$ to be the variables 7 to 17, $\mathbf{Z} = (Z_1, \dots, Z_9)$ to be the variables 18 to 26, and Y to be variable 5 (Midrange Price).

Let S denote the uncorrected sum-of-squares-and-products matrix based on the data for these variables. The fictitious model we shall consider for the prediction game has $(\mathbf{X}, \mathbf{Z}, Y)$ multivariate normal, with mean $\mathbf{0}$ and dispersion matrix $\Sigma = S/n$. The predictive distribution of Y , based on any collection of linear transforms of the X 's and Z 's, will then be normal, with a mean-formula that can be computed by running the zero-intercept sample linear regression of Y on those variables, and variance that will not depend on the values of the predictors. Note that, although our calculations are based on the sample data, the values computed are not estimates, but are the correct values for our fictitious model.

Let U_1 be the variable so obtained from the sample regression Y on $\mathbf{X} \equiv (X_1, \dots, X_{11})$.

$i:$	1	2	3	4	5	6	7	8	9	10	...
$u_i:$	40.62	39.49	39.34	39.51	39.55	39.54	39.66	39.75	39.73917	39.73925	...
$v_i:$	38.28	39.40	39.46	39.54	39.56	39.63	39.67	39.74	39.73924	39.73925	...

Table 1: Sequence of market predictions for Y

Recall that both experts are supposed to know the model, hence Σ , and know which variables each is observing. Consequently both know the form of U_1 , but initially only E_1 , who knows the values of (X_1, \dots, X_{11}) , can compute its value, u_1 say. Since his round-1 forecast for Y is normal with mean u_1 , while its variance is already computable by both experts, the effect of E_1 issuing his forecast is to make the value u_1 of U_1 public knowledge.

It is now E_2 's turn to play. At this point she knows the values of U_1 and (Z_1, \dots, Z_9) , and her forecast is thus obtained from the sample regression of Y on these variables. Let this regression function (computable by both experts) be V_1 ; then at this round E_2 effectively makes the value v_1 of V_1 public.

Now at round 2, E_1 regresses Y on $(X_1, \dots, X_{11}, V_1)$ (U_1 , which is a linear function of his privately known X 's, being redundant), and announces the value u_2 of the computed regression function U_2 . And so on.

The relevant computations are easy to conduct using the statistical software package R (R Development Core Team, 2011). At each stage, we computed the 82 fitted values based on the regression just performed. These can then be used as values for the new predictor variable to be included in the next regression. Moreover, convergence of the forecast sequence will be reflected in convergence of these fitted values. We observe this convergence, both for the fitted values and the predicted standard deviations, from round 10 onwards: as soon as E_1 has access to the values of U_1, \dots, U_9 , he effectively knows Z_1, \dots, Z_9 , and his forecast becomes the same as that based on the pooled data. And as soon as E_1 makes that public, E_2 can make the same forecast.

As a numerical illustration, suppose E_1 has observed

$$\mathbf{X} = \mathbf{x} = (16, 25, 2, 1, 8, 4.6, 295, 6000, 1985, 0, 20.0),$$

and E_2 has observed

$$\mathbf{Z} = \mathbf{z} = (5, 204, 111, 74, 44, 31.0, 14, 3935, 1).$$

Before entering the prediction market, E_1 's point forecast for Y , based on his data $\mathbf{X} = \mathbf{x}$, is $u_1 = 40.6163$, and E_2 's point forecast for Y , based on her data $\mathbf{Z} = \mathbf{z}$, is $v_0 = 30.6316$. If they could combine their data, the forecast, based on $(\mathbf{X}, \mathbf{Z}) = (\mathbf{x}, \mathbf{z})$, would be 39.73925.

On entering the market, the sequence of their predictions is as given in Table 1. We see convergence to the value based on all the data at round 10, by which point E_2 ,

having publicly announced the values of 9 predictor variables, has effectively revealed all her 9-dimensional private information to E_1 . The predictions of both experts will stay the same thereafter.

As a second illustration, suppose E_1 has observed

$$\mathbf{X} = \mathbf{x} = (22, 30, 1, 0, 4, 3.5, 208, 5700, 2545, 1, 21.1),$$

and E_2 has observed

$$\mathbf{Z} = \mathbf{z} = (4, 186, 109, 69, 39, 27.0, 13, 3640, 0).$$

Before entering the prediction market, E_1 's point forecast for Y is 27.80968, and E_2 's point forecast is 36.593865. Their market forecasts converge at round 10 to 31.22983, the forecast based on all the data.

These two cases illustrate within-sequence convergence, but to a random (*i.e.*, data-dependent) limit. \square

A similar example for a linear prediction that gives the same basic results was shown in Dutta and Polemarchakis (2014). They however do not give a numerical illustration.

4.3 Limited consensus

In all the above examples, convergence was either to a vacuous state, or to a complete consensus based on the totality of the pooled private information. As the following example shows, it is also possible to converge to an intermediate state.

Example 5 Suppose θ and X_1 have independent $N(0, 1)$ distributions, while, given (θ, X_1) , $X_2 \sim N(\theta X_1, 1)$. Expert E_1 observes $H_1 = X_1$, while E_2 observes $H_2 = X_2$. The interest is in predicting θ . A sufficient statistic for θ , based on the combined data (X_1, X_2) , is $(X_1 X_2, |X_1|) = (S_1, S_2)$, say. The posterior distribution is

$$\theta \mid (S_1, S_2) = (s_1, s_2) \sim N\left(\frac{s_1}{1 + s_2^2}, \frac{1}{1 + s_2^2}\right).$$

Straightforward computations deliver the joint density of (X_1, X_2) , marginalising over θ :

$$f(x_1, x_2) = (2\pi)^{-1} (1 + x_1^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \left(x_1^2 + \frac{x_2^2}{1 + x_1^2}\right)\right). \quad (3)$$

Because (3) is unchanged if we change the sign of either or both of x_1 and x_2 , we deduce (what may be obvious from the symmetry of the whole set-up):

Proposition 5 *Conditionally on $|X_1|$ and $|X_2|$, $\text{sign}(X_1)$ and $\text{sign}(X_2)$ behave as independent fair coin-flips.*

At the first round, E_1 declares his posterior for θ , based on X_1 —but, since $X_1 \perp\!\!\!\perp \theta$ this supplies no information at all about θ . (So we would get the same answer if E_2 were to go first—the order in which they announce their opinions does not matter.)

Now E_2 goes. Since $X_2 \mid \theta \sim N(0, 1 + \theta^2)$, with sufficient statistic $|X_2|$, E_2 is effectively putting $|X_2|$ into the public pot.

At the start of round 2, E_1 knows X_1 and $|X_2|$. By Proposition 5, $\text{sign}(X_2)$ is still equally likely to be 1 or -1 . So E_1 knows S_2 , but only knows $|S_1|$ —for him, S_1 is either $|S_1|$ or $-|S_1|$, each being equally likely. His posterior is thus a 50–50 mixture of the associated posteriors

$$N\left(\frac{|S_1|}{1 + S_2^2}, \frac{1}{1 + S_2^2}\right)$$

and

$$N\left(\frac{-|S_1|}{1 + S_2^2}, \frac{1}{1 + S_2^2}\right).$$

On E_1 's now announcing this mixture posterior, he is effectively communicating $(|S_1|, S_2) \equiv (|X_1| \times |X_2|, |X_1|)$. The total information in the public pot is thus now equivalent to $(|X_1|, |X_2|)$.

It is now E_2 's turn again. At this point she knows $(|X_1|, X_2)$, so $(|S_1|, S_2)$ —but still does not know $\text{sign}(S_1)$, which again behaves as a coin-flip. Her forecast distribution is thus exactly the same as E_1 's. So we get convergence to the above mixture posterior at round 2. But this limiting forecast is not the same as that based on the pooled data, which would be the relevant single component of the mixture.

Note that, at convergence, the pool of public knowledge is $(|X_1|, |X_2|)$. Since θ has the identical mixture posterior whether conditioned on $(|X_1|, |X_2|)$, on $(X_1, |X_2|)$, or on $(|X_1|, X_2)$, we have both $\theta \perp\!\!\!\perp X_1 \mid (|X_1|, |X_2|)$ and $\theta \perp\!\!\!\perp X_2 \mid (|X_1|, |X_2|)$, in accordance with Proposition 4. \square

It might appear that the above behaviour is highly dependent on the symmetry of the problem, but this is not so. As the following analysis shows, the same limited consensus behaviour arises on breaking the symmetry.

Example 6 Consider the same problem as in Example 5 above, with the sole modification that the prior distribution of θ is now $N(\mu, 1)$, where μ is non-zero. The posterior distribution of θ , based on the full data (X_1, X_2) or its sufficient statistic (S_1, S_2) , is now

$$\theta \mid (S_1, S_2) = (s_1, s_2) \sim \Pi(s_1, s_2) := N\left(\frac{\mu + s_1}{1 + s_2^2}, \frac{1}{1 + s_2^2}\right).$$

The following result is immediate.

Proposition 6 *Given only $|S_1| = m_1, S_2 = m_2$, the posterior distribution is a mixture:*

$$\theta \sim M(m_1, m_2) = \pi(1)\Pi(m_1, m_2) + \pi(-1)\Pi(-m_1, m_2) \quad (4)$$

where

$$\pi(j) = P(\text{sign}(S_1) = j \mid |S_1| = m_1, S_2 = m_2) \quad (j = \pm 1). \quad (5)$$

Proposition 7 *Conditionally on $|X_1|$ and $|X_2|$:*

(i). $\text{sign}(X_1) \perp\!\!\!\perp \text{sign}(X_1 X_2)$

(ii). $\text{sign}(X_2) \perp\!\!\!\perp \text{sign}(X_1 X_2)$

Proof. (i) The joint density of (X_1, X_2) , marginalising over θ , is

$$f(x_1, x_2) = (2\pi)^{-1} (1 + x_1^2)^{-\frac{1}{2}} \exp -\frac{1}{2} \left(x_1^2 + \frac{(x_2 - \mu x_1)^2}{1 + x_1^2} \right).$$

This is unchanged if we change the signs of both x_1 and x_2 . Consequently, given $|X_1| = m_1, |X_2| = m_2$, $P(X_1 = m_1, X_2 = m_2) = P(X_1 = -m_1, X_2 = -m_2)$, while $P(X_1 = m_1, X_2 = -m_2) = P(X_1 = -m_1, X_2 = m_2)$. But this is equivalent to

$$\begin{aligned} P(\text{sign}(X_1) = 1, \text{sign}(X_1 X_2) = 1) &= P(\text{sign}(X_1) = -1, \text{sign}(X_1 X_2) = 1) \\ P(\text{sign}(X_1) = 1, \text{sign}(X_1 X_2) = -1) &= P(\text{sign}(X_1) = -1, \text{sign}(X_1 X_2) = -1). \end{aligned}$$

Thus $P(\text{sign}(X_1) = 1 \mid \text{sign}(X_1 X_2) = 1) = P(\text{sign}(X_1) = 1 \mid \text{sign}(X_1 X_2) = -1) = \frac{1}{2}$, which in particular implies $\text{sign}(X_1) \perp\!\!\!\perp \text{sign}(X_1 X_2)$.

(ii) We have

$$\begin{aligned} P(\text{sign}(X_2) = 1 \mid \text{sign}(X_1 X_2) = 1) &= P(\text{sign}(X_1) = 1 \mid \text{sign}(X_1 X_2) = 1) \\ P(\text{sign}(X_2) = 1 \mid \text{sign}(X_1 X_2) = -1) &= P(\text{sign}(X_1) = -1, \text{sign}(X_1 X_2) = -1) \end{aligned}$$

So from (i), conditional on $|X_1| = m_1, |X_2| = m_2$, $P(\text{sign}(X_2) = 1 \mid \text{sign}(X_1 X_2) = 1) = P(\text{sign}(X_2) = 1 \mid \text{sign}(X_1 X_2) = -1) = \frac{1}{2}$ so that, in particular, $\text{sign}(X_2) \perp\!\!\!\perp \text{sign}(X_1 X_2)$. \square

In the first round, E_1 and E_2 behave exactly as before, and again, at the start of round 2, the public pot contains $|X_2|$. So now E_1 knows X_1 and $|X_2|$. In terms of the sufficient statistic he knows $(|S_1|, S_2)$, but does not know $\text{sign}(S_1)$. Moreover, by Proposition 7(i), his additional knowledge of $\text{sign}(X_1)$ contains no relevant further information about $\text{sign}(S_1)$. Consequently, he will compute and announce the mixture posterior $M(|S_1|, S_2)$. From this it is possible to deduce the values of $|S_1|$ and S_2 . Hence at this point the public pot contains $(|S_1|, S_2)$.

Now E_2 knows $(|S_1|, S_2)$, but is still ignorant of $\text{sign}(S_1)$. And again, although she has the additional knowledge of $\text{sign}(X_2)$, by Proposition 7(ii) this contains no relevant further information about $\text{sign}(S_1)$. Consequently, E_2 will have the same posterior distribution $M(|S_1|, S_2)$, which will be the final (but limited) consensus.

(Note that an essentially identical analysis will hold with any prior distribution for θ .) \square

5 Discussion

We have displayed a variety of behaviours for a process where two experts take it in turns to update their probability of a future event, conditioning only on the revealed probabilities of the other. Although there will always be convergence to a limiting value, this may or may not be the same as what they could achieve if they were able to pool all their private information.

We have supposed throughout that, although each expert may be unaware of the private information held by the other, he does at least know which variables the other expert knows—just not their values. When this cannot be assumed there will be much greater freedom to update one’s own probability on the basis of the revealed probability of the other. Nevertheless this freedom is restricted. Some theory relevant to the case of combining the announced probabilities of a number of experts, without even knowing the private variables on which these are based, may be found in Dawid et al. (1995). It would be challenging, but valuable, to extend this to the present sequential case.

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