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**A BOOTSTRAP BIAS CORRECTION OF LONG RUN FOURTH ORDER
MOMENT ESTIMATION IN THE CUSUM OF SQUARES TEST**

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A Bootstrap Bias Correction of long run fourth order moment estimation in the CUSUM of squares test

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Abstract

The aim of this paper is to propose a bias correction of the estimation of the long run fourth order moment in the CUSUM of squares test proposed by Sansó *et al.* (2004) for the detection of structural breaks in financial data. The correction is made by using the stationary bootstrap proposed by Politis and Romano (1994). The choice of this resampling technique is justified by the stationarity and weak dependence of the time series under the assumptions which ensure the existence of the limiting distribution of the test statistic, under the null hypothesis.

Some Monte Carlo experiments have been implemented in order to evaluate the effect of the proposed bias correction considering two particular data generating processes, the GARCH(1,1) and the log-normal stochastic volatility. The effectiveness of the bias correction has been evaluated also on real data sets.

Keywords: CUSUM of squares test, Structural breaks, Bias correction.

JEL Code: C120, C580, G170.

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1 Introduction

Many studies (see for example Andreou and Ghysels, 2009 and the references therein) agree on the possible existence of structural breaks in the variance of financial time series and on the importance of their detection.

The literature on testing for structural breaks is extensive and a lot of procedures and tests have been proposed (for a review see Perron, 2006). An important role in analysing variance constancy, both in methodological advances and in applications, is played by statistical methods based on the CUSUM of squares test proposed by Brown *et al.* (1975).

The widespread use of these tests is mainly due to their simplicity and to the lack of the need to specify a particular pattern of variation. These tests are non parametric and, as a consequence, they ensure asymptotic invariance against a fairly general class of generating processes. Moreover they do not require any previous knowledge of the timing and the number of the shifts.

These properties make these tests very popular in many empirical applications especially in a financial context, where different conditional heteroskedasticity patterns can be assumed. The original procedure for detecting breaks in the variance has been developed, for independent processes, by Inlan and Tiao (1994). More recently it has been extended to some types of dependent processes; in particular Kokoszka and Leipus (2000) have considered the case of β -mixing processes assuming an ARCH (∞) process.

In a more general framework, starting from a different set of assumptions, Sansó *et al.* (2004) have proposed a test which is similar to Kokoszka and Leipus test. It makes adjustments to the original proposal that allow the time series to obey a wide class of dependent processes, including GARCH and log-normal stochastic volatility processes, under the null.

This test has been widely used in the econometric literature; recently, it has been employed to identify structural breaks in US GDP and oil price growth rates (Gadea *et al.*, 2016), in daily returns of U.S. and BRICS stock markets (Mensi *et al.*, 2016) and in stock return data on two Chinese stock return market indexes (Ni *et al.*, 2016).

Optimal properties of the test statistic are obtained by means of a consistent estimator of the long-run fourth order moment of the considered series, which provides the right standardization to obtain a statistic free of nuisance parameters. A popular strategy, used in the econometric literature and in the original paper, is based on a non parametric kernel method

often referred to as the kernel Heteroskedasticity Autocorrelation Consistent (HAC) estimator.

When using this estimation in the context of long run variance estimation, it is quite common to consider the Bartlett Kernel. However the kernel function depends on an additional parameter, the bandwidth, which controls the degree of complexity of the estimation and, consequently, plays a fundamental role in order to obtain accurate properties of the resultant HAC estimator.

In the econometric literature it is quite common the use of data dependent rules in which this parameter is fixed as a function of the correlation structure of the series. In this context the proposal by Newey and West (1994) is a classical choice. Although other methods, which lead to inconsistent estimation, have been proposed (Kiefer and Vogelsang, 2002; 2005), the Newey and West procedure remains the most used and it will be considered in this paper. Moreover, when using this approach, the test is asymptotically correct in presence of short-run dynamics or conditional heteroskedasticity. On the contrary, in the case of strong dependence and in relative small samples, the HAC estimation of the long run fourth order moment could be inefficient and, as a consequence, significant biases may arise leading to significant size distortion of the test (Rodriguez and Rubia, 2007).

The aim of this paper is to propose a bias correction of the estimation of the long run fourth order moment in the CUSUM of squares test proposed by Sansó *et al.* (2004). The correction is made by using a resampling technique, namely the stationary bootstrap, proposed by Politis and Romano (1994). This choice is justified by the stationarity and weakly dependence of the time series which ensure the existence of the limiting distribution of the test statistic under the null hypothesis. Following Politis and White (2004), the optimal expected block size for the stationary bootstrap has to be fixed to $c * T^{1/3}$ where T is the time series length and c is a constant that has been fixed in a suitable way depending on the specific properties of the data generation process.

The structure of the paper is as follows. Section 2 provides an overview of testing procedures for single and multiple breaks in the unconditional variance. Starting from the first proposal by Inclan and Tiao (1994), in the case of *i.i.d.* Normal data, a correction to the cumulative sum of squares test, proposed by Sansó *et al.* (2004) is discussed and some theoretical results on its limiting distribution are reported. In section 3 the problem of the estimation of the long run fourth order moment is addressed reviewing the most popular approaches and focusing

on the data driven rule. Section 4 introduces the bootstrap approach in the context of dependent data and discusses the bootstrap bias correction of the long run fourth order moment. In particular the stationary bootstrap is illustrated and a procedure to select the optimal expected block size is proposed. In section 5 Monte Carlo experiments are implemented in order to evaluate the effect of the bias correction on the considered CUSUM of squares test. The attention is focused on two particular data generating processes, the GARCH (1,1) and the log-normal stochastic volatility which are the most used models in financial applications. In section 6 the effectiveness of the bias correction is evaluated on a real data set consisting of two financial time series; the IPC Index - Mexico and CNX Nifty Index - India. Some final remarks close the paper.

2 Tests for structural breaks

Let $\{a_t\}$, $t = 1, \dots, T$ denotes the time series of interest and assume, for simplicity, that the unconditional and the conditional mean is 0. The problem of testing the hypothesis that the unconditional variance of a_t is constant could be formulated as:

$$H_0 : \sigma_t^2 = \sigma_a^2 \quad \text{for all } t = 1, \dots, T \quad (1)$$

against the alternative hypothesis of $m + 1$ regimes

$$H_A : \sigma_t^2 = \begin{cases} \sigma_1^2 & \text{for } t = 1, \dots, \tau_1 \\ \sigma_2^2 & \text{for } t = \tau_1 + 1, \dots, \tau_2 \\ \vdots & \quad \quad \quad \vdots \\ \sigma_{m+1}^2 & \text{for } t = \tau_m + 1, \dots, T \end{cases} \quad (2)$$

The number of breakpoints m and their locations τ_1, \dots, τ_m , are assumed to be unknown. The problem will be analysed by considering at first the case of a single change-point and then the extension to the more general case of multiple change-points.

2.1 Testing for a single structural break

The most popular test for a change point is based on the IT statistic proposed by Inclan and Tiao (1994). It is based on cumulative sum of squares statistic and it defined as:

$$IT = \sup_k |(T/2)^{1/2} D_k| \quad (3)$$

where $D_k = (C_k/C_T) - (k/T)$ and $C_k = \sum_{t=1}^k a_t^2$ for $k = 1, \dots, T$. The value of k corresponding to the maximum of $|(T/2)^{1/2} D_k|$ is the estimate of the break date.

Proposition 1 (Inclan and Tiao, 1994) *Under the hypothesis that $\{a_t\}$ are i.i.d. with distribution $N(0, \sigma_a^2)$, the asymptotic distribution of the IT statistic is given by:*

$$IT \xrightarrow{A} \sup_r |W^*(r)| \quad (4)$$

where $W^*(r) \equiv W(r) - rW(1)$ is a Brownian Bridge and $W(r)$ is a standard Brownian motion¹.

Proof The proof follows most of the steps used by Inclan and Tiao (1994).

Let a_1, a_2, \dots be a sequence of independent, identically distributed Normal($0, \sigma_a^2$) and $\xi_t = a_t^2 - \sigma_a^2$ with $E[\xi_t] = 0$ and $\text{Var}(\xi_t) = 2\sigma_a^4$. Let:

$$X_T(r) = \frac{1}{\sqrt{T}\sqrt{2\sigma_a^4}} S_{\lfloor rT \rfloor} + (rT - \lfloor rT \rfloor) \frac{1}{\sqrt{T}\sqrt{2\sigma_a^4}} \xi_{\lfloor rT \rfloor + 1} \quad (5)$$

where $S_T = \xi_1 + \dots + \xi_T$, $r \in [0, 1]$ and $\lfloor \cdot \rfloor$ is the floor function.

By Donsker's theorem (Billingsley 1999, thm. 8.2) $X_T(r) \xrightarrow{\mathcal{D}} W(r)$ and so $X_T(r) - rX_T(1) \xrightarrow{\mathcal{D}} W^*(r)$.

Let $rT = k$, $k = 1, \dots, T$. By simple algebra, it is:

$$\begin{aligned} X_T(r) - rX_T(1) &= \frac{1}{\sqrt{2\sigma_a^2}\sqrt{T}} \left[S_k - \frac{k}{T} S_T \right] + \frac{(rT - \lfloor rT \rfloor)}{\sqrt{2\sigma_a^2}\sqrt{T}} \xi_{\lfloor rT \rfloor + 1} \\ &= \frac{1}{\sqrt{2\sigma_a^2}\sqrt{T}} \left[\sum_{t=1}^k (a_t^2 - \sigma_a^2) - \frac{k}{T} \sum_{t=1}^T (a_t^2 - \sigma_a^2) \right] \frac{(rT - \lfloor rT \rfloor)}{\sqrt{2\sigma_a^2}\sqrt{T}} \xi_{\lfloor rT \rfloor + 1} \end{aligned}$$

¹see Appendix A for definition of Brownian Bridge and Brownian Motion

$$= \frac{1}{\sqrt{2}\sigma_a^2\sqrt{T}} \left[\sum_{t=1}^T a_t^2 \right] D_k + \frac{(rT - \lfloor rT \rfloor)}{\sqrt{2}\sigma_a^2\sqrt{T}} \xi_{\lfloor rT \rfloor + 1}$$

It follows that:

$$\sqrt{\frac{T}{2}} D_k \left[\frac{\frac{1}{T} \sum_{i=1}^T a_i^2}{\sigma_a^2} \right] = X_T(r) - rX_T(1) - \frac{(rT - \lfloor rT \rfloor)}{\sqrt{2}\sigma_a^2\sqrt{T}} \xi_{\lfloor rT \rfloor + 1}$$

As $T \rightarrow \infty$

$$\frac{\frac{1}{T} \sum_{i=1}^T a_i^2}{\sigma_a^2} \rightarrow 1 \quad \text{by the law of large number}$$

and

$$\sup_r \left| \frac{(rT - \lfloor rT \rfloor)}{\sqrt{2}\sigma_a^2\sqrt{T}} \xi_{\lfloor rT \rfloor + 1} \right| \xrightarrow{\mathcal{P}} 0$$

it is

$$\sqrt{T/2} D_k \xrightarrow{\mathcal{D}} W^*(r) \quad (6)$$

Applying the Continuous Mapping Theorem the result is proven. \square

The distribution of $\sup_r |W^*(r)|$ is given in Billingsley (1999); finite-sample critical values for the test can be determined by simulation.

The most serious weakness of the IT statistic is that its asymptotic distribution is strongly dependent on the assumption that the random variables $\{a_t\}$ have a Normal independent and identical distribution. The following proposition generalized the asymptotic distribution defined previously in the case of $\{a_t\} \sim i.i.d(0, \sigma^2)$

Proposition 2 (Sansó et al., 2004) *If $\{a_t\} \sim i.i.d(0, \sigma_a^2)$ and $E(a_t^4) = \eta_4 < \infty$, then:*

$$IT \xrightarrow{A} \sqrt{\frac{\eta_4 - \sigma_a^4}{2\sigma_a^4}} \sup_r |W^*(r)| \quad (7)$$

Proof The proof follows most of the steps used in the proposition 1. Note that, for not mesokurtic random variable $\text{Var}(\xi_t) = \eta_4 - \sigma_a^4$. It is:

$$\sqrt{\frac{2\sigma_a^4}{\eta_4 - \sigma_a^4}} \sqrt{\frac{T}{2}} D_k = \sqrt{\frac{2\sigma_a^4}{\eta_4 - \sigma_a^4}} \sqrt{\frac{T}{2}} \left(\frac{C_k}{C_T} - \frac{k}{T} \right)$$

$$\begin{aligned}
&= \frac{\sqrt{T}\sigma_a^2}{\sqrt{\eta_4 - \sigma_a^4}} \left(\frac{C_k - \frac{k}{T}C_T}{C_T} \right) \\
&= \frac{1}{\sqrt{T}\sqrt{\eta_4 - \sigma_a^4}} \left(C_k - \frac{k}{T}C_T \right) \left(\frac{\sigma_a^2}{\frac{1}{T}C_T} \right) \\
&= \frac{1}{\sqrt{T}\sqrt{\eta_4 - \sigma_a^4}} \left(\sum_{t=1}^k a_t^2 - \frac{k}{T} \sum_{t=1}^T a_t^2 \right) \left(\frac{\sigma_a^2}{\frac{1}{T} \sum_{t=1}^T a_t^2} \right) \\
&= \frac{1}{\sqrt{T}\sqrt{\eta_4 - \sigma_a^4}} \left(\sum_{t=1}^k (a_t^2 - \sigma_a^2) - \frac{k}{T} \sum_{t=1}^T (a_t^2 - \sigma_a^2) \right) \left(\frac{\sigma_a^2}{\frac{1}{T} \sum_{t=1}^T a_t^2} \right) \\
&= \frac{1}{\sqrt{T}\sqrt{\eta_4 - \sigma_a^4}} \left(\sum_{t=1}^k \xi_t - \frac{k}{T} \sum_{t=1}^T \xi_t \right) \left(\frac{\sigma_a^2}{\frac{1}{T} \sum_{t=1}^T a_t^2} \right)
\end{aligned}$$

As $T \rightarrow \infty$, by the law of large number:

$$\frac{\sigma_a^2}{\frac{1}{T} \sum_{t=1}^T a_t^2} \rightarrow 1$$

and, by Donsker's theorem (Billingsley 1999, thm. 8.2), it is:

$$\frac{1}{\sqrt{T}\sqrt{\eta_4 - \sigma_a^4}} \sum_{t=1}^{\lfloor rT \rfloor} \xi_t \xrightarrow{\mathcal{D}} W(r) \quad (8)$$

From the previous results, it follows that:

$$\begin{aligned}
\sqrt{\frac{2\sigma_a^4}{\eta_4 - \sigma_a^4}} \sqrt{\frac{T}{2}} D_k &\xrightarrow{\mathcal{D}} W(r) - rW(1) \\
&\equiv W^*(r)
\end{aligned}$$

and so:

$$\sqrt{T/2} D_K \xrightarrow{\mathcal{D}} \sqrt{\frac{\eta_4 - \sigma_a^4}{2\sigma_a^4}} W^*(r) \quad (9)$$

Applying the Continuous Mapping Theorem the result is proven. \square

Remark 1 If $\{a_t\}$ is a Gaussian process, $\eta_4 = 3\sigma_a^4$ and the result in proposition 1 holds. When $\eta_4 < 3\sigma_a^4$ the distribution is platykurtic and the test could be too conservative. On the contrary, if $\eta_4 > 3\sigma_a^4$ the distribution is leptokurtic and, as a consequence, the effective size of the IT test could be greater than the nominal one

Hence, the asymptotic distribution is not free of nuisance parameters and, when using the critical values of the supremum of a Brownian Bridge, size distortion should be expected.

Despite these problems, the IT test has been used for detecting changes in the volatility of financial time series (see Aggarwal *et al.*, 1999; Huang and Yang, 2001). However its reliance on the Gaussian distribution is problematic since financial data are known to be non Gaussian and can exhibit heavy tailed behaviour. However, as it will become apparent later, adjustments can be applied to IT statistic that allow $\{a_t\}$ to obey a wide class of dependent processes, including GARCH processes, under the null. In this direction, two different tests, K_1 and K_2 , which consider the fourth moment properties of the disturbances and the conditional heteroskedasticity, have been proposed. The K_1 test corrects for non-mesokurtosis. It is defined as:

$$K_1 = \sup_r |T^{-1/2} B_k| \quad (10)$$

where:

$$B_k = \frac{C_k - (k/T)C_T}{\sqrt{\widehat{\eta}_4 - \widehat{\sigma}_a^4}} \quad (11)$$

with $\widehat{\eta}_4 = T^{-1} \sum_{t=1}^T a_t^4$ and $\widehat{\sigma}_a^2 = T^{-1} C_T$

Proposition 3 (Sansó *et al.*, 2004) *Under the hypothesis that $\{a_t\} \sim i.i.d(0, \sigma_a^2)$ and $E(a_t^4) < \infty$ the asymptotic distribution of the K_1 statistic is given by*

$$K_1 \xrightarrow{A} \sup_r |W^*(r)| \quad (12)$$

Proof The proof follows immediately from Proposition 2. Indeed, from (9) it is:

$$\sqrt{\frac{2\sigma_a^4}{\eta_4 - \sigma_a^4}} \sqrt{\frac{T}{2}} D_K \xrightarrow{D} W^*(r) \quad (13)$$

By simple algebra and by using $\widehat{\sigma}_a^2 = T^{-1} C_T$, it is easy to show that:

$$\begin{aligned} \sqrt{\frac{2\sigma_a^4}{\eta_4 - \sigma_a^4}} \sqrt{\frac{T}{2}} D_K &= \frac{1}{\sqrt{T} \sqrt{\eta_4 - \sigma_a^4}} \left[C_k - \frac{k}{T} C_T \right] \\ &\equiv T^{-1/2} B_k \end{aligned} \quad (14)$$

Applying the Continuous Mapping Theorem the result is proven. \square

The statistics IT and K_1 are both based on the assumption of independence of the sequence of random variables and, as a consequence, they are not suitable for financial time series. To overcome this problem, Sansó *et al.* (2004) have proposed to correct the cumulative sum of squares with an estimation of the persistence. However some conditions on $\{a_t\}$ are required.

Assumption 1 $E(a_t) = 0$ and $E(a_t^2) = \sigma_a^2 < \infty \forall t \geq 1$

Assumption 2 $\sup_t E(|a_t|^{\psi+\epsilon}) < \infty$ for some $\psi \geq 4$ and $\epsilon > 0$

Assumption 3 $\omega_4 = \lim_{T \rightarrow \infty} E(T^{-1}(\sum_{t=1}^T (a_t^2 - \sigma_a^2))^2)$

Assumption 4 $\{a_t\}$ is α -mixing with coefficients α_j satisfying:

$$\sum_{j=1}^{\infty} \alpha_j^{(1-2/\psi)} < \infty$$

Assumption 1 imposes that the unconditional mean of the process is 0 and the unconditional variance exists and it is constant. The latest is the hypothesis which has to be tested.

Assumption 2 states the existence of moments of order greater than four although it does not impose them a constant value and, as a consequence, some sort of non stationarity is allowed.

Assumption 3 establishes a value for ω_4 which can be interpreted as the long run fourth order moment of $\{a_t\}$.

Assumption 4 allows for temporal dependence in the process $\{a_t\}$ (see appendix B for definition of mixing processes) so that, although there may be substantial dependence among recent event, events which are separated by long intervals of time are almost independent.

From the previous assumptions, it is evident that the existence of the fourth order moment and the finiteness of the long-run fourth order moment are both necessary conditions to derive the asymptotic distribution of the test statistic. Therefore, classes of processes which do not satisfy these conditions cannot be considered. For example, conditions 2 and 3 are not fulfilled if $\{a_t\}$ is a sequence of random variables independent and identically distributed as a t-Student with three degrees of freedom.

Under the previous assumptions, Sansó *et al.* (2004) have proposed the following statistic:

$$K_2 = \sup_k |T^{-1/2}G_k| \quad (15)$$

where $G_k = \widehat{\omega}_4^{-1/2}[C_k - (k/T)C_T]$ and $\widehat{\omega}_4$ is a consistent estimator of ω_4 .

The asymptotic distribution of the three statistics IT , K_1 and K_2 under the variance-persistence processes are illustrated in the following proposition.

Proposition 4 (Sansó *et al.*, 2004) *Under the assumptions 1-4, the asymptotic distribution of the statistics are given by*

$$\begin{aligned} a) \quad IT &\xrightarrow{A} \sqrt{\frac{\omega_4}{2\sigma_a^4}} \sup_r |W^*(r)| \\ b) \quad K_1 &\xrightarrow{A} \sqrt{\frac{\omega_4}{\eta_4 - \sigma_a^4}} \sup_r |W^*(r)| \\ c) \quad K_2 &\xrightarrow{A} \sup_r |W^*(r)| \end{aligned}$$

Proof The proof follows most of the step used by Sansó *et al.* (2004). For the assumption 4, $\{a_t\}$ are α -mixing and then they are also ξ_t . Thus it is possible to use the Herrndorf's theorem (Herrndorf, 1984) being assumptions 1-4 a restrictive case of the conditions of the theorem.

Let us consider:

$$\begin{aligned} \frac{1}{\sqrt{T}\sqrt{\omega_4}} \left(C_k - \frac{k}{T}C_T \right) &= \frac{1}{\sqrt{T}\sqrt{\omega_4}} \left(\sum_{t=1}^k \xi_t - \frac{k}{T} \sum_{t=1}^T \xi_t \right) \\ &\xrightarrow{D} W(r) - rW(1) \\ &\equiv W^*(r) \end{aligned}$$

Suppose that $\widehat{\omega}_4$ is a consistent estimator of ω_4 . For the previous result:

$$\begin{aligned} \frac{1}{\sqrt{T}}G_K &= \frac{1}{\sqrt{T}}(\widehat{\omega}_4)^{-1/2} \left(C_k - \frac{k}{T}C_T \right) \\ &\xrightarrow{D} W^*(r) \end{aligned}$$

Applying the Continuous Mapping Theorem the result c) is proven.

Given that

$$\frac{1}{\sqrt{T}} \left(C_k - \frac{k}{T} C_T \right) \xrightarrow{\mathcal{D}} (\widehat{\omega}_4)^{1/2} W^*(r) \quad (16)$$

by simple algebra:

$$\begin{aligned} \sqrt{\frac{T}{2}} D_k &= \sqrt{\frac{T}{2}} \left(\frac{C_k}{C_T} - \frac{k}{T} \right) \\ &= \frac{1}{\sqrt{T}} \left(C_k - \frac{k}{T} C_T \right) \frac{1}{\sqrt{2}} \frac{T}{C_t} \\ &\xrightarrow{\mathcal{D}} \sqrt{\frac{\omega_4}{2\sigma_a^4}} W^*(r) \end{aligned}$$

Applying the Continuous Mapping Theorem the result a) is proven.

Moreover it is:

$$\begin{aligned} \frac{1}{\sqrt{T}} B_k &= \frac{1}{\sqrt{T}} \frac{1}{\sqrt{\widehat{\eta}_4 - \widehat{\sigma}_a^4}} \left(C_k - \frac{k}{T} C_T \right) \\ &\xrightarrow{\mathcal{D}} \sqrt{\frac{\omega_4}{\eta_4 - \sigma_a^4}} W^*(r) \end{aligned}$$

Applying the Continuous Mapping Theorem the result b) is proven. \square

2.2 Testing for multiple structural breaks

Many of the procedures for analysing single break point can be adapted to the analysis of multiple break points by recursively applying a single change point method. Different approaches have been proposed in the statistical and the econometric literature, such as the ICCS algorithm (Inclan and Tiao, 1994), the Binary Segmentation (Edwards and Cavallisforza, 1965), the Segmented Neighborhood search (Braun *et al.*, 2000) and the Minimum description method (Davis *et al.*, 2006). Here, the attention has been focused on the binary segmentation which is based on successive application of the test to sub-series obtained consecutively after a change-point is found. The procedure starts by applying the detection method to the whole series. If no change-point is found the procedure is stopped; otherwise the data are split into two segments and the detection method is applied to each of them. The procedure is repeated until no further change-points are detected.

The choice of binary segmentation algorithm is justified by its simplicity and efficiency; it is

very fast and it could be implemented with a low computational cost.

However, the procedure could produce spurious break points because of the presence of extreme observations which can be erroneously interpreted as being change points (Ross, 2013). To partially solve this problem and to better identify the break points location, it is necessary to implement a pruning procedure. In this paper a pruning procedure, in the spirit of the ICCS algorithm (Inclan and Tiao, 1994), has been suggested.

In the case that m breaks have been detected at times $\tau_1, \tau_2, \dots, \tau_m$, with $\tau_0 = 1$ and $\tau_{m+1} = T$, the pruning procedure can be implemented as follows.

- The detection method is applied to the segment $[\tau_{i-1}, \tau_{i+1}]$ for $i = 1, \dots, m$
- If no change-point is found in the segment $[\tau_{i-1}, \tau_{i+1}]$, the break at τ_i is not considered as change point. If a new change point is detected, it replaces the old one at τ_i .
- The procedure is repeated until the number of change points does not change and the points found in each new step are "close" to those on the previous step.

The main problem, when a detection method is applied with a searching algorithm, is that the use of the same critical value for any segments may distort the performance of the iterative procedure. To overcome this problem, it is possible to use the response surfaces methodology (MacKinnon, 1994). This approach allows to approximate the asymptotic distribution of a test and to obtain critical values for the statistic that converges to this distribution. This method is based on the determination of the following regression:

$$q_T^\alpha = \sum_{j=1}^d \theta_{p_j}^\alpha T^{p_j} + v_T \quad (17)$$

where q_T^α is the α -quantile of the test statistic for a sample size T , θ_{p_j} are the parameters to be estimated and the regressors are powers of T . The values of q_T^α could be obtained by using Monte Carlo experiments for different values of sample size T . These empirical quantiles are then used in the response surface regression to calculate the estimated parameters. For the K_2 test statistics. Sansó *et al.* (2004) reported the final estimates of the response surface with $\alpha = 0.05$.

$$\hat{q}_T^{0.05} = 1.405828 - 3.317278 * T^{-0.5} + 31.22133 * T^{-1} - 1672.206 * T^{-2} + \quad (18)$$

$$+ 52870.53 * T^{-3} - 411015 * T^{-4}$$

The parameter estimates are all significant, the fit can be considered quite good, being $R^2 = 0.999$, and the residual standard deviation is very small ($\hat{\sigma}_v = 0.013$).

The equation (18) can be used to compute the critical values of the K_2 test for different sample sizes T .

3 Long-run fourth order moment estimation

When using the statistic K_2 , it is necessary to obtain a consistent estimator of ω_4 the long-run fourth order moment of a_t . Since it provides the right standardization to obtain a statistic free of nuisance parameters, the determination of this estimator plays a crucial role in implementing the CUSUM of squares test.

The long-run fourth order moment of a_t can be rewritten in terms of the long-run variance of the zero mean random variable $\xi_t = a_t^2 - \sigma_a^2$ and, as a consequence, ω_4 can be estimated by using the spectral density of ξ_t at frequency zero. One possibility is to use a parametric approach based on the Akaike estimator of the spectrum. However, a more popular strategy is based on a non parametric kernel method often referred to as the kernel Heteroskedasticity Autocorrelation Consistent (HAC) estimator. It is defined as:

$$\hat{\omega}_4 = \hat{\gamma}_0 + 2 \sum_{l=1}^m [w(l, m)] \hat{\gamma}_l \quad (19)$$

where

$$\hat{\gamma}_l = T^{-1} \sum_{t=l+1}^T (a_t^2 - \hat{\sigma}_a^2)(a_{t-l}^2 - \hat{\sigma}_a^2) \quad (20)$$

$w(\cdot)$ is a suitable kernel weighting function and m is a truncation lag or bandwidth parameter. The kernel function, which is assumed to verify the standard conditions of continuity, squared integrability and symmetry, ensures non-negative estimates. In the context of long run variance estimation, it is quite common, to consider the Bartlett Kernel. It is defined as:

$$w(l, m) = 1 - \frac{l}{m + 1} \quad (21)$$

The choice of the bandwidth parameter m plays a fundamental role in order to obtain accurate properties of the resultant HAC estimator. It captures the covariance structure of the underlying series; if the temporal dependence increases, a larger value of m is necessary to account for a larger number of non-zero covariances.

From an asymptotic point of view, this parameter is a function of the series length T , $m = m_T$ and it should verify $m_T \rightarrow \infty$ as $T \rightarrow \infty$ in such a way that $m_T/T \rightarrow 0$, *i.e.*, $m_T = o(T)$. Under very general assumptions on the data generating process, this approach provides consistent estimates.

In empirical applications, different procedures to choose m_T have been proposed based on deterministic rules or on data dependent methods.

In a first proposal, the bandwidth depends only on the sample size T ; it can be fixed by setting $m_T = [bT^a]$ where a and b are suitable constants such that $b > 0$ and $0 < a < 1$. A common choice is $m_T = [4(T/100)^{2/9}]$ (see Rodrigues and Rubia, 2007).

The second approach assumes that the bandwidth can be chosen as a function of the correlation structure of the series and thus it results data dependent (see Andrews, 1991; Newey and West, 1994). In the case of Bartlett kernel, the optimal bandwidth is defined as:

$$m_T = \min \left\{ T, \lfloor \theta T^{\frac{1}{3}} \rfloor \right\} \quad (22)$$

where

$$\theta = 1.447(S_1^2/S_0^2)^{\frac{1}{3}} \quad (23)$$

$$S_j = \hat{\gamma}_0 + 2 \sum_{i=1}^{m_T^*} i^j \hat{\gamma}_i \quad j = 0, 1 \quad (24)$$

and m_T^* is a given deterministic pre-bandwidth parameter.

More recently, in a different context, inconsistent long run variance estimators have been proposed (Kiefer and Vogelsang, 2002; 2005). They are based on a bandwidth verifying the condition $m_T/T \rightarrow b$, where b is a fixed constant such that $0 < b \leq 1$. This is equivalent to fix the bandwidth as $m_T = bT$ where b represents the proportion of sample autocovariances included in the estimation. Under this approach $m_T = O_P(T^2)$ and, as a consequence, the

HAC estimator is not consistent.

A common choice is to fix $b = 1$ (Kiefer and Vogelsang, 2002); in this case $m_T = T$ and so the estimator uses the entire sample. Although such estimators are inconsistent, when they are used in test statistics, the resulting asymptotic approximation seems to have some advantages in improving the size properties compared to the conventional approach. However, as pointed out by Cai and Shintani (2006) for unit-root testing, the power of the test could decrease significantly.

In the CUSUM framework, this problem has been addressed by Rodriguez and Rubia (2007). They have found that when the data generating process is a GARCH (1,1) model, the CUSUM test seems to over-reject the null hypothesis when a consistent estimator of ω_4 is used and to under-reject in the case of inconsistent estimator. Moreover they have found a significant distortion of the test resulting from the long run variance estimation in small sample and, even in large samples, when the empirical process exhibits persistent volatility.

All the previous results confirm the important role of the estimation of the long run fourth order moment to perform the CUSUM test. Unfortunately, even if a consistent estimator is considered and although inference based on the resulting test statistic is asymptotically valid, it is well known that in finite sample HAC estimator could have a significant bias which is evident not only in samples of small dimension but also in large samples, especially when the data generating process exhibits persistent volatility.

In this paper we propose a bias correction of the estimation of ω_4 based on the use of a resampling technique, namely the stationary bootstrap. With this approach we will be able to reduce the bias of the estimator of the long run fourth order moment and, as a consequence, the size distortion of the CUSUM test.

4 The bootstrap bias correction

The bootstrap is an approach to approximate the distribution of a statistic of interest and to assign measures of accuracy to statistical estimates. This technique involves resampling the original data to form many pseudo-time series and re-estimating the statistic, or the quantity of interest, on each bootstrap sample.

Starting from the pioneering Efron's paper (Efron, 1979) in the context of independent and

identically distributed data, many bootstrap procedures have been proposed in the literature, in order to solve more complex statistical problems.

In the context of dependent data, a natural approach is to model explicitly the dependence in the data through a parametric model; the bootstrap sample is then drawn from the fitted model. Such model-based approach is inconsistent if the model used for resampling is misspecified. Alternatively, nonparametric model free bootstrap schemes can be used.

In particular the block bootstrap scheme is one of the most widely used bootstrap methods in time series analysis. In this scheme, blocks of consecutive observations are resampled randomly with replacement, from the original series and assembled by joining the blocks together in random order in order to obtain a simulated version of the original series. Therefore this resampling technique maintains the dependence structure of the data within a pseudo sample.

There are basically two different ways of proceeding, depending on whether the blocks are overlapping or non-overlapping. Both approaches have been introduced by Hall (1985) in the context of spatial data. For univariate time series, Carlstein (1986) has proposed non-overlapping blocks, whereas Künsch (1989) has suggested overlapping blocks.

The block bootstrap method does not require the selection of a model but it is sensitive to the block length specification, which is the crucial point of the procedure. If the blocks are long, a poor estimate of the distribution of the statistic could be obtained whereas too short blocks could not preserve, in the resampled series, the original dependence in the data.

Several rules have been suggested to select the optimal length of the blocks; they are based on different criteria but often they are useful only as rough guides. However, unless the length of the series is long enough to allow long and numerous blocks, the dependence structure may be difficult to preserve, especially when the data have a complex and long range dependence structure. In such cases, the block bootstrap scheme tends to generate resampled series that are less dependent than the original ones. Moreover by construction, in the block bootstrap method, the bootstrapped time series has a nonstationary (conditional) distribution. Politis and Romano (1994) have introduced a variant of the block bootstrap, namely the stationary bootstrap, that is generally applicable for stationary weakly dependent time series.

Unlike the block bootstrap method, where the block length is fixed, in the stationary bootstrap method the length of each block of observations is generated from a geometric distribution with parameter Q . As a consequence, the pseudo time series generated by this procedure is

actually stationary and so, the stationarity property of the original series is replicated in the resample time series (Politis and Romano, 1994).

In this paper this resampling technique has been used to obtain a bias correction of the estimator of the long run fourth order moment (defined in equation 19) in the case of a strictly stationary and weakly dependent process. In this context, given the real number $0 < Q < 1$, the procedure runs as follows:

Step 1. Randomly select an observation, say a_t , from the data as the first bootstrapped observation a_1^* .

Step 2. With probability Q , a_2^* is set to a_{t+1} , the observation following the previously sampled observation, and with probability $1 - Q$, the second bootstrapped observation a_2^* is randomly selected from the original data.

Step 3. Repeat recursively step 2 to form (a_1^*, \dots, a_T^*) , the bootstrap series with T observations.

In this scheme, the block size is the (random) number of bootstrapped observations that are drawn consecutively. The block size follows a geometric distribution with parameter Q , and so the bootstrapped results depend on the choice of this parameter. The average length of a block is $1/Q$, where Q is the parameter of the geometric distribution. Thus, $1/Q$ should play the same role as the block length parameter in the block bootstrap. Of course, when Q approaches zero, the resulting stationary bootstrap is like the *i.i.d.* bootstrap. However, as pointed out by Politis and Romano (1994), the stationary bootstrap is less sensitive to the choice of Q than the block bootstrap is to the choice of the block length. Politis and White (2004) have suggested that, under very general conditions, the optimal expected block size for the stationary bootstrap is $c * T^{1/3}$ where c is a constant depending on the specific properties of the data generation process and on the problem at hand. In our case, the constant c has been fixed equal to θ defined in equation (23).

For the long run fourth order moment, the bootstrap bias correction could be obtained by a Monte Carlo procedure which has been implemented as follows.

Step 4. Generate B different bootstrap series $(a_{1,b}^*, \dots, a_{T,b}^*)$ for $b = 1, \dots, B$. For each generated series calculate the long run fourth order moment $(\widehat{\omega}_{4,1}^*, \dots, \widehat{\omega}_{4,B}^*)$.

Step 5. Estimate the bias:

$$\widehat{bias} = \frac{1}{B} \sum_{b=1}^B \widehat{\omega}_{4,b}^* - \widehat{\omega}_4 \quad (25)$$

Step 6. The bias corrected estimator of the long run fourth order moment is:

$$\widehat{\omega}_4^{bc} = \widehat{\omega}_4 - \widehat{bias} = 2\widehat{\omega}_4 - \frac{1}{B} \sum_{b=1}^B \widehat{\omega}_{4,b}^* \quad (26)$$

5 Monte Carlo Experiment

In this section results from a set of Monte Carlo simulation experiments have been reported and discussed. The aim is to evaluate the performance of the K_2 test in which the estimation of the long run fourth order moment has been bias corrected using the stationary bootstrap. In particular, we compare our proposal with that obtained by using a classical HAC estimation. We considered two alternative data generating processes, namely GARCH(1,1) with Normal error and log-Normal stochastic volatility. In the next two subsections we review the conditions necessary to guarantee that the processes match the assumptions 1-4 under which the test has been developed.

5.1 The data generating process: GARCH (1,1)

The GARCH(1,1) model, proposed by Bollerslev (1986), is the most commonly applied parameterization in empirical econometrics; it is a very parsimonious model and usually it is adequate to obtain good performances in terms of fitting and forecasting (Hansen and Lunde, 2005).

The canonical GARCH(1,1) process $(a_t)_{t \in \mathbb{Z}}$ with volatility process $(\sigma_t)_{t \in \mathbb{Z}}$ is defined as:

$$a_t = \sigma_t \epsilon_t \quad t \in \mathbb{Z} \quad (27)$$

with

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad \alpha_0 > 0 \quad \alpha_1, \beta_1 \geq 0 \quad (28)$$

where $\{\epsilon_t\}$ is a sequence of *i.i.d.* random variables with mean zero and unit variance. The conditions on α_0 , α_1 and β_1 ensure that the conditional variance σ_t^2 is positive.

The probabilistic proprieties of the GARCH (1,1) process are well known (for a review see Lindner, 2009). In the following the questions of stationarity², moment conditions and strong mixing proprieties will be reviewed.

Nelson (1990) has proved that the GARCH(1,1) process with $\alpha_0, \alpha_1, \beta_1 > 0$ is strictly stationary if and only if

$$-\infty < E(\ln(\beta_1 + \alpha_1 \epsilon_t^2)) < 0 \quad (29)$$

By the Jensen inequality and under the assumption that $\{\epsilon_t\}$ has unit variance, it is easy to show that $\alpha_1 + \beta_1 < 1$ is a sufficient condition for (29). Moreover Bollerslev (1986) has stated that the GARCH(1,1) process is weakly stationary if and only if $\alpha_1 + \beta_1 < 1$. Bougerol and Picard (1992) have also shown that every strictly stationary GARCH(1,1) process is ergodic. Carrasco and Chen (2002) have proved that, under very general conditions, if the process is strictly stationary then a_t and σ_t^2 are β -mixing³ with geometric rate. Because of the β -mixing implies α -mixing, the processes a_t and σ_t^2 are also α -mixing with geometric rate.

In conclusion, the condition $\alpha_1 + \beta_1 < 1$ is necessary and sufficient for weakly stationarity, sufficient for strictly stationarity and ergodicity and it ensures that the process is β -mixing with geometric rate.

Once the stationarity and mixing conditions are defined, it is important to define conditions for the existence and the finiteness of the moments of order higher than two. For the GARCH(1,1) model a necessary and sufficient condition has been given by Bollerslev (1986) for Normal innovation and by He and Teräsvirta (1999a; 1999b) for general noise sequence. In particular, if ϵ_t is standard Normally distributed, the condition for the existence of the fourth order moment is

$$3\alpha_1^2 + 2\alpha_1\beta_1 + \beta_1^2 < 1 \quad (30)$$

²When considering a stationary GARCH(1,1) process, the stationarity is referred to the bivariate process $(a_t, \sigma_t)_{t \in \mathbb{N}_0}$.

³Mixing conditions describe some types of asymptotic independence which allow the use of suitable limit theorems. The common types of mixing conditions have been introduced in Appendix B; for an extensive treatment see Doukhan (1994).

This condition is clearly more restrictive than $\alpha_1 + \beta_1 < 1$ and so it ensures all the previous properties.

5.2 The data generating process: Log-Normal Stochastic Volatility

The canonical log-normal Stochastic Volatility (SV) process $(a_t)_{t \in \mathbb{Z}}$ with volatility process $(\sigma_t)_{t \in \mathbb{Z}}$ is defined (Taylor, 2007) as:

$$a_t = \sigma_t \epsilon_t \quad t \in \mathbb{Z} \quad (31)$$

$$\ln(\sigma_t^2) = \phi_0 + \phi_1 \ln(\sigma_{t-1}^2) + v_t \quad \phi_0, \phi_1 > 0 \quad (32)$$

where $\{\epsilon_t\}$ is a sequence of *i.i.d.* random Normal variables with mean zero and unit variance, $\{v_t\}$ is a sequence of *i.i.d.* random Normal variables with mean zero and variance σ_v^2 , $\{\epsilon_t\}$ and $\{v_t\}$ are mutually independent.

Denoting $h_t = \ln(\sigma_t^2)$ the models (31) and (32) can be rewritten as:

$$a_t = \exp\left(\frac{1}{2}h_t\right) \epsilon_t \quad (33)$$

$$h_t = \phi_0 + \phi_1 h_{t-1} + v_t \quad (34)$$

This model is the natural discrete-time approximation to the continuous-time Orstein-Uhlenbeck process used in finance theory.

As pointed out by Harvey *et al.* (1994) if $|\phi_1| < 1$, the process h_t is ergodic and stationary both in the strict and weak sense. It is trivial to show that its mean is $\mu_h = \phi_0/(1 - \phi_1)$ and its variance is $\sigma_h^2 = \sigma_v^2/(1 - \phi_1^2)$.

The condition $|\phi_1| < 1$ ensures also the stationarity of a_t being the product of two stationary processes. Moreover the condition is sufficient by itself to ensure the existence of the moments. The odd moments of a_t are all zero because ϵ_t is symmetric; the even moments can be obtained by using standard results, being $\exp(h_t)$ lognormal. In particular, it is easy to show that the variance of a_t is equal to $\exp(\mu_h + \frac{1}{2}\sigma_h^2)$ and, unlike a GARCH model, the fourth order moment always exists when h_t is stationary.

Finally, Carrasco and Chen (2002) have shown that if a_t is a stationary process it is also β -mixing and so the previous results apply.

5.3 Simulation design and results

In order to analyse the false rejection rates, data generating processes without structural breaks have been considered. In particular for the GARCH(1,1) process, we have considered a parametric space which includes empirical values typically observed in practice. In particular, we have fixed $\alpha_0 = 0.1$ and we have set $\alpha_1 \in \{0.05, 0.075, 0.01\}$ and β_1 in such a way that $\alpha_1 + \beta_1 \in \{0.7, 0.8, 0.9, 0.95\}$. For the SV model we have set $\phi_1 \in \{0.7, 0.8, 0.9, 0.95\}$, $\sigma_v \in \{0.1, 0.2, 0.3\}$ and $\phi_0 = -\frac{1}{2} \frac{\sigma_v}{1+\phi_1}$, so that the unconditional variance of a_t is equals to 1.

To evaluate small samples as well as asymptotic behaviour of the analysed test, four different sample sizes $T \in \{500, 1000, 2000, 4000\}$ have been considered.

The simulations are based on 1000 Monte Carlo runs and, in the bootstrap implementation, the number of replications has been fixed equal to 999. The nominal level of the test is 0.05. As pointed out previously, the expected block size in the stationary bootstrap scheme should be fixed to $c * T^{1/3}$ where c is a suitable constant. In this experiment, we have proposed to fix c according to the optimal bandwidth parameter of the Bartlett kernel used in the HAC estimation of the long run fourth order moment. More precisely, c is set to θ defined in equation (23). The value of the expected block size are reported in table 1 for the GARCH (1,1) specification and in table 2 for the log-normal stochastic volatility process. As expected, the values increase with the sample size and with the persistence.

The empirical false rejection rates for the different specifications of the GARCH (1,1) process are reported in table 3. The entries in bold indicate the values inside the asymptotic acceptance interval at level 0.95. ⁴

It is quite evident that for all specifications of the parameters the false rejection rates of the test when a bootstrap bias correction is used are always less than the corresponding value obtained with the HAC estimator.

The test with HAC estimation has high values of the false rejection rate for small sample size ($T = 500$ and $T = 1000$); for large sample sizes reasonable rates are achieved only in the case of time-series with not extremely persistent conditional heteroskedasticity ($\alpha_1 + \beta_1 \leq 0.8$).

When the proposed bootstrap bias correction is used, for all sample sizes the false rejection

⁴The asymptotic acceptance interval at 95% is defined as: $\hat{p} \pm 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{T}}$ where \hat{p} is fixed at nominal value 0.05.

tion rates could be considered not statistically different from the nominal value when the persistence is not high ($\alpha_1 + \beta_1 \leq 0.8$). For large sample size, also for high persistence ($\alpha_1 + \beta_1 = 0.9$) the values are inside the acceptance interval.

The previous results are confirmed in the case of log-normal stochastic volatility model (table 4); the bootstrap bias correction always reduces the false rejection rates for all the parameter specifications and for all the sample sizes. However the test with HAC estimation generates false rejection rates not significantly different from the nominal value for $\phi_1 = 0.7$, in the case of small and moderate sample size ($T = 500$, $T = 1000$ and $T = 2000$) and for $\phi_1 \leq 0.8$ in the case of high sample size ($T = 4000$). Again, the false rejection rates achieve the nominal value, in the case of the test with bootstrap bias correction, for $\phi_1 \leq 0.8$, when the sample size is small and moderate ($T = 500$, $T = 1000$ and $T = 2000$) and for $\phi_1 \leq 0.9$ in the case of high sample size ($T = 4000$).

In order to analyse if the test is able to identify a break present in the data generating process, we have generated time series with a single structural break located in the middle of the sample. In this case three different sample sizes $T \in \{1000, 2000, 4000\}$ have been fixed. The variance in the first of the two regimes, identified by the structural breaks, has been fixed to 1 while the variance in the second to $\{2, 2.5, 3\}$. For the GARCH (1,1) process the parameters are $\alpha_1 = 0.1$ and $\beta_1 \in \{0.7, 0.8, 0.9, 0.95\}$; for the log-normal stochastic volatility model we have set $\phi_1 \in \{0.7, 0.8, 0.9, 0.95\}$ and $\sigma_v = 0.3$. With respect to the previous simulation design, the parameters α_1 and σ_v have been fixed since the performances of the test with HAC estimation and with the bootstrap bias correction seem to be not sensitive to the choice of these parameters.

For both models the constant has been fixed in such a way that the variances in the regimes assume the specified values. The simulations are based on 1000 Monte Carlo runs; the bootstrap parameters have been fixed as in the previous analysis. For each sample size, the relative frequency of zero break, a single break and more than one break have been computed for both the long run fourth order moment estimations. In this simulation we have also examined if the procedure is able to well identify the location of the break. To this aim it is assumed that a break is correctly located if it is identified in an interval of length 40 around the exact location. When the procedure identifies more than 1 break, it has been also computed the relative frequency of exact location of "at least one" of the identified breaks.

In table 5 the results for the GARCH (1,1) process are reported. In general, it is quite evident

that the bootstrap approach is able to better identify the presence of one structural break in the data for all the specifications of the parameters and for all the sample sizes. Moreover, when one break is identified, the test with the bias correction produces a high relative frequency of exact location. The differences between the two approaches seem to be more clear when the persistence and the sample size increase. By looking at the case in which more than one breaks are identified, it is evident that the bootstrap procedure is more conservative, in the sense that the relative frequency is lower in all the considered simulation schemes. Furthermore, the zero break relative frequency tends to 0 as the sample size increases highlighting not significant differences between the two long run fourth order estimations.

The previous results are confirmed in the case of log-normal stochastic volatility model (table 6). In particular, the two approaches seem to have equal performances for $\phi_1 = 0.70$; for the other values of the parameter ϕ_1 , the test with the bootstrap bias correction seems to significantly outperform that obtained with the HAC estimator, especially when the sample size increases.

6 An example on real data

In order to evaluate the effectiveness of the proposed procedure, an example on real data has been implemented.

The series are the IPC index (Índice de Precios y Cotizaciones) which is the broadest indicator of the overall performance of the Mexican Stock Exchange and the CNX Nifty index which is the National Stock Exchange of India's benchmark stock market index for Indian equity market.

The data are obtained from the Oxford-Man Institute's realised library and the sample covers the period from January 3, 2000 to May 20, 2016 for the IPC Index and from January 1, 2003 to May 20, 2016 for the CNX Nifty index.

In order to investigate the presence of break points in the unconditional variance of both the series, an analysis based on the K_2 test with the HAC estimator of the long run variance and the proposed bootstrap bias correction has been employed. In both cases, a binary segmentation with the pruning method described in section 2 has been used.

For both the series, the identified break points with the two procedures are reported in table 7 and 8 respectively for the IPC-Mexico and CNX Nifty-India; in figure 1 and 2 the two-

standard-deviation bands for each of the regimes defined by the structural breaks are shown. For the IPC index, the procedure based on HAC estimation identifies seven break points. Most of them could be considered as spurious false positive due to the presence of extreme observations which correspond to very small period of high variability. When applying the bootstrap bias correction to the estimation of the long run fourth order moment, the number of the break points has reduced. In particular, the proposed procedure identifies a single break point located at July 2009. The reduction of the volatility after this break point could be due to the effects of the end of the global financial crisis. Moreover the location of the break is consistent with that identified by the test based on HAC estimation; the difference between the two break dates is not relevant.

For the CNX Nifty Index both procedures identify three break points located in the same dates. The first break occurs at November 2007 which can be considered as the starting point of the global financial crisis. The second break point is located, as in the previous case, in July 2009 that could be assumed as the end of the financial crisis. The last break point is located at May 2012 and it is characterized by a reduction of volatility. In this case both procedures seem to work quite well and to be robust to extreme observations. The results of this toy example confirm that the test based on bootstrap estimation seems to be less sensitive to the extreme values and, as a consequence, more conservative with respect to the classical HAC estimation.

7 Concluding remarks

In this paper we have proposed a bias correction of the estimation of the long run fourth order moment in the CUSUM of squares test proposed by Sansó *et al.* (2004). It is a development of the Inclan and Tiao test (1994) for the detection of changes in unconditional variance suitable for financial time series analysis.

The correction has been made by using the stationary bootstrap which has been justified by the stationarity and the weak dependence of the time series under the assumptions which ensure the existence of the limiting distribution of the test statistic under the null. After a brief introduction and discussion on the CUSUM of squares test and on the bootstrap approach in the context of dependent data, a procedure to select the optimal expected block size has been proposed.

In order to evaluate the effects of the proposed bias correction, Monte Carlo experiments have been implemented focusing on two particular data generating processes, the GARCH(1,1) and the log-normal stochastic volatility models. The results give evidence that the bootstrap approach is better able to correctly identify the presence of structural breaks in the data. More specifically, in the GARCH(1,1) data generating process without breaks, the false rejection rates could be considered non statistically different from the nominal values when the persistence is not high and for large sample size even in the case of high persistence. These results are confirmed in the case of the log-normal stochastic volatility model.

When the data are generated assuming the presence of a single break in the middle of the sample, the bias corrected estimation is able to correctly identify the break and its location, for all the sample sizes and in both the considered models.

The proposed procedure has been applied to analyse the presence of structural breaks in two real time series, IPC-Mexico and CNX Nifty-India. In both cases it seems to work quite well resulting more robust with respect extreme observations.

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8 Figures and Tables

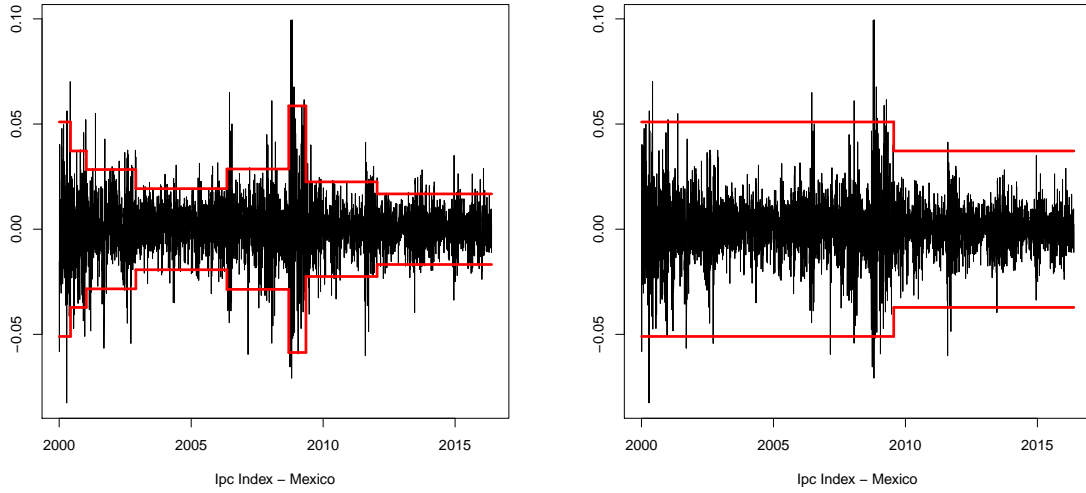


Figure 1: IPC Index-Mexico.

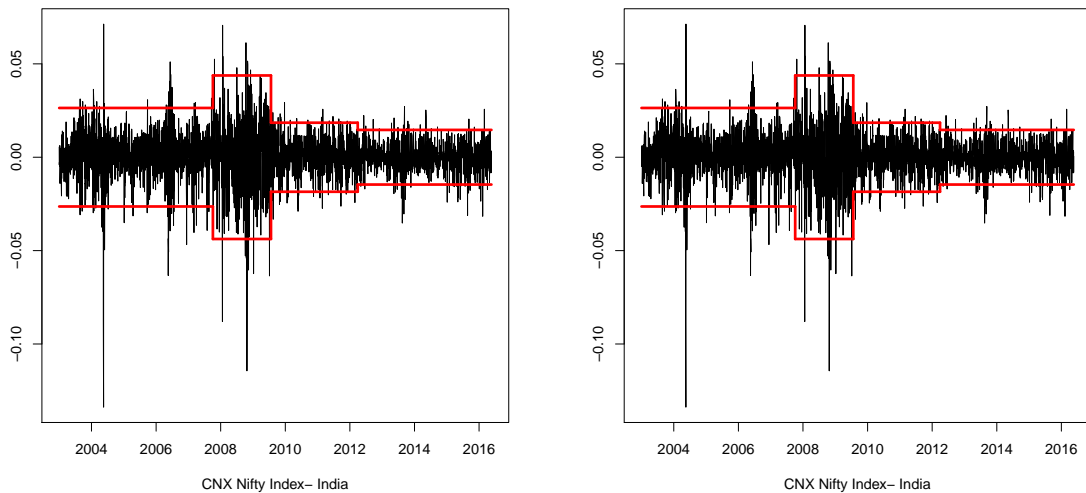


Figure 2: CNX Nifty Index-India.

Note: The red lines are two-standard-deviation bands for the regimes defined by the structural breaks identified by the binary segmentation and the pruning procedure. On the left the breaks have been identified by the K_2 test with the HAC estimator of the long run variance; on the right with the proposed bootstrap bias correction.

Table 1: Expected block size of the stationary bootstrap for the GARCH (1,1) data generating process.

α_1	$\alpha_1 + \beta_1$	T=500	T=1000	T=2000	T=4000
0.05	0.70	6.51	8.26	10.52	14.17
	0.80	7.11	9.69	13.34	19.26
	0.90	8.09	12.02	17.47	26.80
	0.95	8.89	13.90	20.48	32.15
0.075	0.70	7.20	9.65	12.83	17.46
	0.80	8.18	11.72	16.44	23.63
	0.90	9.63	14.64	21.06	31.78
	0.95	10.77	16.70	24.23	37.29
0.10	0.70	7.92	10.93	14.75	20.04
	0.80	9.16	13.35	18.63	26.65
	0.90	10.91	16.41	23.37	34.97
	0.95	12.12	18.44	26.46	40.24

Table 2: Expected block size of the stationary bootstrap for the log-normal stochastic volatility data generating process.

σ_v	ϕ_1	T=500	T=1000	T=2000	T=4000
0.10	0.70	6.09	7.11	7.96	9.89
	0.80	6.21	7.23	8.00	10.81
	0.90	6.44	8.13	10.63	16.64
	0.95	7.44	11.13	16.82	26.73
0.20	0.70	6.17	7.24	8.73	11.33
	0.80	6.46	8.42	11.25	16.27
	0.90	8.58	13.18	19.21	28.92
	0.95	11.51	17.43	25.13	38.06
0.30	0.70	6.22	8.06	10.39	13.77
	0.80	7.85	11.04	15.69	22.16
	0.90	11.09	16.43	23.06	34.34
	0.95	13.21	19.33	27.30	40.92

Table 3: GARCH(1,1) data generating process: False Rejection Rate for the K_2 test with the HAC estimator of the long run variance and with the bootstrap bias correction.

α_1	$\alpha_1 + \beta_1$	T=500		T=1000		T=2000		T=4000	
		HAC	BOOT	HAC	BOOT	HAC	BOOT	HAC	BOOT
0.05	0.70	0.073	0.057	0.066	0.055	0.056	0.045	0.057	0.047
	0.80	0.083	0.062	0.078	0.063	0.064	0.049	0.057	0.047
	0.90	0.149	0.097	0.126	0.087	0.100	0.071	0.066	0.054
	0.95	0.268	0.199	0.235	0.166	0.205	0.141	0.140	0.090
0.075	0.70	0.072	0.053	0.068	0.057	0.056	0.045	0.055	0.047
	0.80	0.085	0.062	0.076	0.055	0.065	0.046	0.051	0.046
	0.90	0.156	0.103	0.133	0.083	0.100	0.068	0.069	0.047
	0.95	0.284	0.206	0.254	0.155	0.197	0.126	0.133	0.077
0.10	0.70	0.076	0.050	0.065	0.047	0.055	0.044	0.050	0.045
	0.80	0.088	0.059	0.075	0.051	0.066	0.044	0.050	0.045
	0.90	0.154	0.093	0.127	0.073	0.096	0.058	0.072	0.044
	0.95	0.281	0.186	0.242	0.136	0.179	0.113	0.122	0.070

Note: The nominal level of the test is 0.05. In bold the values inside the asymptotic acceptance interval at 95%.

Table 4: Log-normal stochastic volatility data generating process: False Rejection Rate for the K_2 test with the HAC estimator of the long run variance and with the proposed bootstrap bias correction.

σ_v	ϕ_1	T=500		T=1000		T=2000		T=4000	
		HAC	BOOT	HAC	BOOT	HAC	BOOT	HAC	BOOT
0.10	0.70	0.057	0.052	0.057	0.053	0.053	0.052	0.054	0.052
	0.80	0.071	0.062	0.059	0.056	0.055	0.051	0.054	0.052
	0.90	0.096	0.085	0.081	0.066	0.084	0.070	0.068	0.056
	0.95	0.205	0.165	0.186	0.155	0.176	0.142	0.133	0.096
0.20	0.70	0.053	0.051	0.058	0.056	0.056	0.053	0.053	0.049
	0.80	0.068	0.062	0.069	0.056	0.065	0.051	0.059	0.052
	0.90	0.135	0.103	0.116	0.084	0.117	0.083	0.077	0.054
	0.95	0.290	0.192	0.240	0.155	0.198	0.127	0.127	0.077
0.30	0.70	0.054	0.041	0.055	0.044	0.056	0.049	0.054	0.047
	0.80	0.086	0.057	0.068	0.049	0.073	0.051	0.050	0.047
	0.90	0.154	0.093	0.127	0.073	0.096	0.058	0.061	0.048
	0.95	0.260	0.145	0.188	0.110	0.139	0.076	0.118	0.060

Note: The nominal level of the test is 0.05. In bold the values inside the asymptotic acceptance interval at 95%.

Table 5: Relative frequency of zero break, a single break and more than one break for the GARCH(1,1) model generated with a break in the middle on the sample size. The values σ_1^2 and σ_2^2 indicate the unconditional variance in the two regimes identified by the break. The entries in brackets are the relative frequencies of exact location (inside an interval of length 40) of "at least one" of the identified breaks.

GARCH(1,1)	T=1000						T=2000						T=4000					
	HAC		BOOT		HAC		BOOT		HAC		BOOT		HAC		BOOT			
	0	1	> 1	0	1	> 1	0	1	> 1	0	1	> 1	0	1	> 1			
$\alpha_1 = 0.1 \beta_1 = 0.60$																		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.0$	0.00	0.89 [0.86]	0.11 [0.10]	0.00	0.92 [0.88]	0.08 [0.08]	0.00	0.89 [0.86]	0.11 [0.10]	0.00	0.91 [0.88]	0.09 [0.08]	0.00	0.90 [0.87]	0.10 [0.09]	0.00		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.5$	0.00	0.87 [0.86]	0.13 [0.12]	0.00	0.90 [0.89]	0.10 [0.09]	0.00	0.88 [0.87]	0.12 [0.11]	0.00	0.91 [0.90]	0.09 [0.09]	0.00	0.89 [0.88]	0.11 [0.10]	0.00		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 3.0$	0.00	0.86 [0.86]	0.14 [0.14]	0.00	0.91 [0.91]	0.09 [0.09]	0.00	0.87 [0.87]	0.13 [0.12]	0.00	0.91 [0.90]	0.09 [0.09]	0.00	0.90 [0.90]	0.10 [0.09]	0.00		
$\alpha_1 = 0.1 \beta_1 = 0.70$																		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.0$	0.01	0.86 [0.81]	0.13 [0.12]	0.01	0.90 [0.84]	0.09 [0.09]	0.00	0.88 [0.81]	0.12 [0.11]	0.00	0.91 [0.84]	0.09 [0.08]	0.00	0.89 [0.84]	0.11 [0.9]	0.00		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.5$	0.00	0.85 [0.83]	0.15 [0.15]	0.00	0.89 [0.87]	0.11 [0.11]	0.00	0.86 [0.85]	0.14 [0.13]	0.00	0.90 [0.88]	0.10 [0.10]	0.00	0.88 [0.86]	0.12 [0.12]	0.00		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 3.0$	0.00	0.84 [0.83]	0.16 [0.15]	0.00	0.89 [0.88]	0.11 [0.11]	0.00	0.86 [0.85]	0.14 [0.13]	0.00	0.91 [0.90]	0.09 [0.09]	0.00	0.88 [0.88]	0.12 [0.11]	0.00		
$\alpha_1 = 0.1 \beta_1 = 0.80$																		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.0$	0.03	0.77 [0.65]	0.20 [0.17]	0.06	0.82 [0.70]	0.12 [0.11]	0.01	0.79 [0.66]	0.20 [0.16]	0.01	0.88 [0.72]	0.11 [0.10]	0.01	0.82 [0.70]	0.18 [0.15]	0.00		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.5$	0.00	0.76 [0.71]	0.24 [0.21]	0.01	0.85 [0.78]	0.14 [0.13]	0.00	0.79 [0.73]	0.21 [0.19]	0.00	0.87 [0.80]	0.13 [0.12]	0.00	0.83 [0.78]	0.17 [0.15]	0.00		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 3.0$	0.00	0.74 [0.72]	0.26 [0.24]	0.01	0.83 [0.80]	0.17 [0.15]	0.00	0.78 [0.76]	0.21 [0.20]	0.00	0.87 [0.84]	0.13 [0.11]	0.00	0.83 [0.79]	0.17 [0.16]	0.00		
$\alpha_1 = 0.1 \beta_1 = 0.85$																		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.0$	0.13	0.54 [0.41]	0.33 [0.28]	0.19	0.62 [0.45]	0.19 [0.16]	0.05	0.61 [0.42]	0.34 [0.24]	0.05	0.75 [0.51]	0.19 [0.13]	0.06	0.70 [0.47]	0.29 [0.18]	0.01		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.5$	0.04	0.58 [0.48]	0.38 [0.33]	0.07	0.68 [0.55]	0.25 [0.21]	0.01	0.64 [0.50]	0.35 [0.28]	0.01	0.76 [0.59]	0.23 [0.17]	0.01	0.71 [0.56]	0.29 [0.23]	0.00		
$\sigma_1^2 = 1.0 - \sigma_2^2 = 3.0$	0.01	0.59 [0.52]	0.40 [0.36]	0.03	0.70 [0.61]	0.27 [0.25]	0.01	0.61 [0.52]	0.38 [0.31]	0.01	0.75 [0.64]	0.24 [0.20]	0.01	0.70 [0.60]	0.30 [0.25]	0.00		

Table 6: Relative frequency of zero break, a single break and more than one break for the log-normal stochastic volatility model generated with a break in the middle on the sample size. The values σ_1^2 and σ_2^2 indicate the unconditional variance in the two regimes identified by the break. The entries in brackets are the relative frequencies of exact location (inside an interval of length 40) of "at least one" of the identified breaks.

Log-Normal Stochastic Volatility	T=1000						T=2000						T=4000						
	HAC		BOOT		HAC		BOOT		HAC		BOOT		HAC		BOOT				
	0	1	> 1	0	1	> 1	0	1	> 1	0	1	> 1	0	1	> 1	0	1	> 1	
$\phi_1 = 0.7 \sigma_v = 0.3$																			
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.0$	0.00	0.91 [0.91]	0.09 [0.09]	0.00	0.92 [0.92]	0.08 [0.08]	0.00	0.90 [0.89]	0.10 [0.09]	0.00	0.91 [0.91]	0.09 [0.08]	0.00	0.91 [0.90]	0.09 [0.09]	0.00	0.91 [0.91]	0.09 [0.08]	0.09 [0.08]
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.5$	0.00	0.91 [0.90]	0.09 [0.09]	0.00	0.92 [0.91]	0.08 [0.08]	0.00	0.90 [0.89]	0.10 [0.09]	0.00	0.92 [0.91]	0.08 [0.08]	0.00	0.91 [0.90]	0.09 [0.08]	0.00	0.92 [0.91]	0.08 [0.07]	0.08 [0.07]
$\sigma_1^2 = 1.0 - \sigma_2^2 = 3.0$	0.00	0.91 [0.91]	0.09 [0.08]	0.00	0.92 [0.91]	0.08 [0.08]	0.00	0.89 [0.89]	0.11 [0.10]	0.00	0.90 [0.90]	0.10 [0.09]	0.00	0.90 [0.90]	0.10 [0.09]	0.00	0.91 [0.91]	0.09 [0.09]	0.09 [0.09]
$\phi_1 = 0.8 \sigma_v = 0.3$																			
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.0$	0.00	0.85 [0.84]	0.15 [0.14]	0.00	0.89 [0.88]	0.11 [0.11]	0.00	0.86 [0.85]	0.14 [0.13]	0.00	0.90 [0.88]	0.10 [0.10]	0.00	0.89 [0.88]	0.11 [0.09]	0.00	0.91 [0.89]	0.09 [0.08]	0.09 [0.08]
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.5$	0.00	0.87 [0.85]	0.13 [0.12]	0.00	0.91 [0.89]	0.09 [0.08]	0.00	0.87 [0.86]	0.13 [0.12]	0.00	0.90 [0.88]	0.10 [0.10]	0.00	0.88 [0.84]	0.12 [0.12]	0.00	0.89 [0.86]	0.11 [0.10]	0.11 [0.10]
$\sigma_1^2 = 1.0 - \sigma_2^2 = 3.0$	0.00	0.84 [0.84]	0.16 [0.15]	0.00	0.89 [0.89]	0.11 [0.10]	0.00	0.85 [0.84]	0.15 [0.14]	0.00	0.89 [0.88]	0.11 [0.10]	0.00	0.89 [0.87]	0.11 [0.09]	0.00	0.92 [0.89]	0.08 [0.07]	0.08 [0.07]
$\phi_1 = 0.9 \sigma_v = 0.3$																			
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.0$	0.13	0.70 [0.55]	0.17 [0.15]	0.21	0.70 [0.55]	0.10 [0.08]	0.02	0.83 [0.63]	0.15 [0.11]	0.03	0.87 [0.67]	0.10 [0.07]	0.00	0.87 [0.64]	0.13 [0.09]	0.00	0.92 [0.67]	0.08 [0.06]	0.08 [0.06]
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.5$	0.03	0.77 [0.67]	0.20 [0.18]	0.06	0.81 [0.71]	0.13 [0.12]	0.00	0.83 [0.70]	0.17 [0.14]	0.00	0.88 [0.75]	0.12 [0.10]	0.00	0.83 [0.68]	0.17 [0.13]	0.00	0.89 [0.73]	0.11 [0.08]	0.11 [0.08]
$\sigma_1^2 = 1.0 - \sigma_2^2 = 3.0$	0.01	0.77 [0.70]	0.22 [0.20]	0.02	0.82 [0.75]	0.16 [0.14]	0.00	0.80 [0.71]	0.20 [0.18]	0.00	0.87 [0.78]	0.13 [0.11]	0.00	0.84 [0.71]	0.16 [0.12]	0.00	0.89 [0.76]	0.11 [0.07]	0.11 [0.07]
$\phi_1 = 0.95 \sigma_v = 0.3$																			
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.0$	0.18	0.56 [0.38]	0.26 [0.20]	0.30	0.56 [0.38]	0.14 [0.11]	0.06	0.65 [0.39]	0.29 [0.20]	0.10	0.73 [0.44]	0.17 [0.12]	0.01	0.75 [0.45]	0.24 [0.14]	0.01	0.85 [0.50]	0.14 [0.08]	0.14 [0.08]
$\sigma_1^2 = 1.0 - \sigma_2^2 = 2.5$	0.27	0.48 [0.31]	0.25 [0.18]	0.40	0.49 [0.32]	0.11 [0.08]	0.15	0.61 [0.34]	0.24 [0.15]	0.23	0.62 [0.35]	0.15 [0.09]	0.03	0.75 [0.39]	0.22 [0.11]	0.05	0.83 [0.43]	0.12 [0.06]	0.12 [0.06]
$\sigma_1^2 = 1.0 - \sigma_2^2 = 3.0$	0.18	0.54 [0.38]	0.28 [0.20]	0.29	0.56 [0.39]	0.15 [0.12]	0.07	0.63 [0.39]	0.30 [0.20]	0.11	0.72 [0.4]	0.17 [0.10]	0.01	0.75 [0.44]	0.24 [0.14]	0.01	0.84 [0.48]	0.15 [0.09]	0.15 [0.09]

Table 7: Volatility breaks dates for IPC Index - Mexico.

IPC Index - Mexico	
HAC	BOOT
2000-01-03	2000-01-03
2000-06-06	-
2001-01-11	-
2002-11-27	-
2006-05-10	-
2008-09-12	-
2009-05-08	2009-07-23
2012-01-18	-
2016-05-20	2016-05-20

Note: On the left the breaks have been identified by the K_2 test with the HAC estimator of the long run variance; on the right with the proposed bootstrap bias correction. In both cases, the binary segmentation and the proposed pruning procedure have been used.

Table 8: Volatility breaks dates for CNX Nifty Index - India.

CNX Nifty Index - India	
HAC	BOOT
2003-01-01	2003-01-01
2007-10-05	2007-10-05
2009-07-23	2009-07-23
2012-03-30	2012-03-30
2016-05-20	2016-05-20

Note: On the left the breaks have been identified by the K_2 test with the HAC estimator of the long run variance; on the right with the proposed bootstrap bias correction. In both cases, the binary segmentation and the proposed pruning procedure have been used.

Appendix

A. Brownian Motion and Brownian Bridge

Definition (Brownian motion) A stochastic process $\{W(t)\}_{t>0}$ is a Brownian Motion (or Wiener Process) if the following properties hold:

1. $W(0) = 0$.
2. For $t_1 < t_2$, $W(t_2) - W(t_1) \sim N(0, t_2 - t_1)$.
3. For any $t_1 < t_2 < \dots < t_K$, the random variables $W(t_k) - W(t_{k-1})$ for $k = 2, 3 \dots K$ are independent.
4. $W(t)$ is continuous in t with probability 1.

Remark 2 For any t_1, \dots, t_K , the random vector $(W(t_1), \dots, W(t_K))$ has a multivariate Normal distribution with 0 mean and $Cov(W(t), W(s)) = \min(t, s)$.

Definition (Brownian Bridge) A stochastic process $\{W^*(t)\}_{t \in [0,1]}$ is a Brownian Bridge if the following properties hold:

1. $W^*(0) = W^*(1) = 0$.
2. For any t_1, \dots, t_K , the random vector $(W^*(t_1), \dots, W^*(t_K))$ has a multivariate Normal distribution with 0 mean.
3. $Cov(W^*(t), W^*(s)) = \min(t, s) - st$.
4. $W^*(t)$ is continuous in t with probability 1.

Remark 3 If $\{W(t)\}_{t>0}$ is a Brownian Motion, then $W^*(r) = W(r) - rW(1)$ is a Brownian Bridge. This property can be proven by showing that $W^*(r)$ satisfies the definition of Brownian Motion.

B. Stationarity and mixing

Definition (Strict Stationarity) The stochastic process $\{Y_t\}_{t \in \mathbb{Z}}$ is **strictly stationary**, if for all $k \in \mathbb{N}$, $h \in \mathbb{Z}$ and $(t_1, \dots, t_k) \in \mathbb{Z}^k$

$$(Y_{t_1}, Y_{t_2}, \dots, Y_{t_k}) \stackrel{d}{=} (Y_{t_1+h}, Y_{t_2+h}, \dots, Y_{t_k+h}) \quad (35)$$

where $\stackrel{d}{=}$ denotes equality in distribution.

Definition (Weak Stationarity) The stochastic process $\{Y_t\}_{t \in \mathbb{Z}}$ is **weakly stationary** or **covariance stationary** if, for all $h, t \in \mathbb{Z}$:

$$E(Y_t) = \mu \quad (36)$$

$$\text{Cov}(Y_t, Y_{t+h}) = \gamma(h) \quad (37)$$

with $\gamma(0) < \infty$

Remark 4 *If the process $\{Y_t\}_{t \in \mathbb{Z}}$ is strictly stationary then it is also weakly stationary if and only if $\text{Var}(Y_t) < \infty$*

Definition (Mixing) Let $\{Y_t\}_{t \in \mathbb{Z}}$ be a strictly stationary process defined on a probability space (Ω, \mathcal{F}, P) . Denote by $\mathcal{F}_{-\infty}^0$ the σ -algebra generated by $(Y_s : s \leq 0)$ and by \mathcal{F}_t^∞ the σ -algebra generated by $(Y_s : s \geq t)$ and for $k \in \mathbb{N}$ let:

$$\alpha_k = \sup_{C \in \mathcal{F}_{-\infty}^0, D \in \mathcal{F}_t^\infty} |P(C \cap D) - P(C)P(D)| \quad (38)$$

$$\beta_k = \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |P(C_i \cap D_j) - P(C_i)P(D_j)| \quad (39)$$

where in the definition of β_k the supremum is taken over all pairs of finite partitions $\{C_1, \dots, C_I\}$ and $\{D_1, \dots, D_J\}$ of Ω such that $C_i \in \mathcal{F}_{-\infty}^0$ and $D_j \in \mathcal{F}_t^\infty$ for each i and j . The coefficients α_k and β_k are the α -mixing coefficients and β -mixing coefficients.

$(Y_t)_{t \in \mathbb{Z}}$ is called **α -mixing** or **(strongly mixing)** if $\lim_{k \rightarrow \infty} \alpha_k = 0$.

$(Y_t)_{t \in \mathbb{Z}}$ is called **β -mixing** or **(absolutely regular)** if $\lim_{k \rightarrow \infty} \beta_k = 0$.

$(Y_t)_{t \in \mathbb{Z}}$ is called **α -mixing with geometric rate** if α_k decay at exponential rate i.e. there are constants $\lambda \in (0, 1)$ and c such that $\alpha_k \leq c\lambda^k$ for every k .

$(Y_t)_{t \in \mathbb{Z}}$ is called **β -mixing with geometric rate** if β_k decay at exponential rate i.e. there are constants $\lambda \in (0, 1)$ and c such that $\beta_k \leq c\lambda^k$ for every k .

Remark 5 Since $\alpha_k \leq \frac{1}{2}\beta_k$, *β -mixing implies α -mixing*