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DIPARTIMENTO DI ECONOMIA

THE MATHEMATICAL FRAMEWORK UNDERLYING THE "SCENARIOS" APPROACH FOR DERIVATE TRANSACTIONS BY ITALIAN LOCAL AUTHORITIES

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THE MATHEMATICAL FRAMEWORK UNDERLYING THE "SCENARIOS" APPROACH FOR DERIVATE TRANSACTIONS BY ITALIAN LOCAL AUTHORITIES

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The mathematical framework underlying the "scenarios" approach for derivate transactions by Italian local Authorities.

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Abstract

The Proposal issued on September, 2009 by the Italian Ministry of Economy and Finance (MEF) for the regulation of derivatives that can be subscribed by Italian local Authorithies outlines the information that should be provided in the contracts in order to assure an approriate level of disclosure. The Proposal has introduced an approch based on so-called "scenarios" requiring the comparison between the corresponding proababilities. We present some comments on the mathematical framework underlying the "scenarios" approach for derivate transactions by Italian local Authorities, showing that it is suited to treat – in a mathematical consistent way – the joint distribution at maturity of the value of the original liability ("*portafoglio finanziario iniziale*") and of the liability after the subscription of the derivative by the local Authorithy ("*portafoglio strutturato*") only for a very particular set of cases.

1 Introduction

On September 22, 2009 the Italian Ministry of Economy and Finance (MEF), in collaboration with the Bank of Italy and the Italian Financial Market Authority (Consob), has issued a proposal[1] (hereafter the Proposal) for the regulation of

[1]Ministero delleconomia e delle finanze, Dipartimento del tesoro, Direzione II e Direzione IV, "Schema di regolamento ministeriale di attuazione dellarticolo 62 del decreto legge 25 giugno 2008, n. 112, convertito con modificazioni dalla legge 6 agosto 2008, n. 133, come sostituito dallarticolo 3 della legge 22 dicembre 2008, n. 03", documento in consultazione, 22 settembre 2009 (www.dt.tesoro.it).

derivatives and financial contracts embedding derivatives that can be subscribed by Italian local Authorithies (*enti territoriali e locali*).
The Proposal outlines (*i*) the tipology of derivatives contracts that can be subscribed, (*ii*) the tipology of derivatives that can be embedded in funding operations, and (*iii*) the information that should be provided in the contracts in order to assure an approriate level of disclosure.
Concerning the last issue, when considering the economic suitability of the use of derivatives in local authorithies debt reorganization, the Proposal has introduced an approch based on so-called "scenarios" requiring the comparison between the corresponding proababilities in case the reorganization would or would not take place.
We present some comments on the mathematical framework underlying the "scenarios" approach for derivate transactions by Italian local Authorities.

2 The scenarios approach

According to the methodological instructions specified by the Proposal ("Istruzioni metodologiche") at par. III of the *Allegato*, the comparison between the *portafoglio finanziario iniziale* and the *portafoglio strutturato* should be done considering the value at maturity of the two portfolios.
Since, at the time the comparison is done, the value at maturity of each portfolio is a random variable, the Proposal requires a comparison between two distribution functions.
To perform such a comparison the Proposal requires the construction of a table reporting the probabilities of the following three scenarios:

1. **first scenario** – in the first scenario the value at maturity of the *portafoglio strutturato* is **smaller** than the value at maturity of the *portafoglio finanziario iniziale*;

2. **second scenario** – in the second scenario the value at maturity of the *portafoglio strutturato* is **in line with** the value at maturity of the *portafoglio finanziario iniziale*;

3. **third scenario** – in the third scenario the value at maturity of the *portafoglio strutturato* is **larger** than the value at maturity of the *portafoglio finanziario iniziale*.

For ease of notation we should call the three probabilities P_L, P_M and P_H. The three scenarios are reported in Table 1, which mimicks a similar table reported at par. III of the Proposal.

Table 1: The three different scenarios (*scenari di valutazione comparativa*) with computational rules (*regola di calcolo*) for the determination of the corresponding probabilities (P). The table mimicks a similar table at par III of the *Allegato*.

SCENARI DI VALUTAZIONE COMPARATIVA DEL PORTAFOGLIO FINANZIARIO STRUTTURATO	P	REGOLA DI CALCOLO
"Il valore del portafoglio finanziario struturato è <u>inferiore</u> a quello del portafoglio finanziario iniziale"	P_L	$F_Y\left(q^X_{\alpha_L}\right)$
"Il valore del portafoglio finanziario struturato è <u>in linea</u> con quello del portafoglio finanziario iniziale"	P_M	$F_Y\left(q^X_{\alpha_H}\right) - F_Y\left(q^X_{\alpha_L}\right)$
"Il valore del portafoglio finanziario struturato è <u>superiore</u> a quello del portafoglio finanziario iniziale"	P_H	$1 - F_Y\left(q^X_{\alpha_H}\right)$

Remark 1. While "is larger" or "is smaller" is precisely defined in mathematical terms, "to be in line" is not mathematically well defined. If the three scenarios are required to be *mutually exclusive* (*i.e.* the occurrence of any one of the three automatically implies the non-occurrence of the other two) and *exahustive*, so that the corresponding probabilities sum up to 1, then "to be in line" should necessarly mean "to be equal". Any other definition of "being in line" would imply that either:

a) the probabilities of the three scenarios do not sum up to 1, or
b) according to the definition of "being in line" (to be given) one should redefine what is meant by "being smaller" and "being larger". ■

In fact, after having defined the above three scenarios, the Proposal specifies – at par. IV/d – how to compute P_L, P_M and P_H and thus what is meant by "to be in line", "to be smaller", "to be larger". In the following we rephrase these definitions.

2.1 Mathematical preliminaries

Let us call X the value of the original liability ("*portafoglio finanziario iniziale*") and Y the value of the liability after the subscription of the derivative by the local Authorithy ("*portafoglio strutturato*"). X and Y are random variables assuming values in $\mathbb{R}$ with distribution functions (*funzioni di ripartizione*) $F_X(x)$ and $F_Y(y)$. By definition, distribution functions are defined over $\mathbb{R}$. However, they are constant outside the support of the random variable: for example if X is bounded to be strictly positive, then $F(x) = 0$ for $x \leq 0$. In more complex cases they might be stepwise constant (*e.g.* in the case of the Cantor distribution).

The joint distribution function of (X, Y) will be denoted $F_{XY}(x, y)$; thus $F_X(x)$ and $F_Y(y)$ are the *marginal* distributions of $F_{XY}(x, y)$.

Moreover, let's recall that for any random variable V (discrete or continuous) with distribution function F, the α-*quantile* of F, also called the *quantile of order* α, is defined as that value $q_\alpha \in \mathbb{R}$ such that

$$F[V < q_\alpha] \leq \alpha \leq F[V \leq q_\alpha] \tag{1}$$

with $\alpha \in (0, 1)$. Special cases are obtained for $\alpha = 1/2$ (the median), $\alpha = 1/4$ (the first *quartile*) and $\alpha = 3/4$ (the third *quartile*).

An important point about quantiles is that they always exists but they are not necessary unique. This is due to the fact that if V is discrete or $F(v)$ is not *strictly* continuous, then a value $\bar{q}$ of the random variables for which $F(\bar{q}) = \alpha$ might not exist.

For continuous random variables the α–quantile of F is expressed using the *generalised inverse* $F^{\leftarrow}$ of the distribution function F:

$$q_\alpha(F) = F^{\leftarrow}(\alpha) := \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}. \tag{2}$$

with the convention that the infimum of an empty set is ∞. If F is continuous and *strictly* increasing then $F^{\leftarrow}$ is the ordinary inverse of F, denoted F^{-1}, so that

$$q_\alpha = F^{-1}(\alpha).$$

2.2 Computing the scenarios' probabilities

According to par. IV/d the probabilities of the three scenarios – P_L, P_M, P_H – should be computed following a three steps procedure:

S1. Find the values x_L and x_H such that the probability of the corresponding lower and upper tails of F_X are 2.5% each.

Remark to S1. As already remarked, these values may not be unique. In fact, they are unique only for a continuous and *strictly* increasing distribution function. For simplicity sake, we could assume that the legislator has implicitely assumed that this is the case, as it holds for the more popular financial models. ■

In the following we let $x_L = q^X_{\alpha_L}$ and $x_H = q^X_{\alpha_H}$, $q^X_{\alpha_{L,H}}$ being the two quantiles of X corresponding to $\alpha_L = 2.5\%$ and $\alpha_H = 97.5\%$.

S2. Compute $F_Y(q^X_{\alpha_L})$ and $F_Y(q^X_{\alpha_L})$ using the distribution of Y.

Remark to S2. This is non-trivial for **a limited set** of random variables Y, *i.e.* those for which $q^X_{\alpha_L}$ and $q^X_{\alpha_L}$ belong to the support of Y. Differently, one has $P_L = 0$ ($P_H = 0$). In this case – in our opinion – the relevance of P_L (P_H) is spoiled as much as $q^X_{\alpha_L}$ ($q^X_{\alpha_H}$) are distant from the lower (upper) bound of Y. We believe this case to be relevant since the use of interest rate swaps to reorganize a floating rate debt in a fixed rate debt is a very common financial practice. ∎

S3. Set:

$$\begin{aligned} P_L &= F_Y(q^X_{\alpha_L}), \\ P_M &= F_Y(q^X_{\alpha_H}) - F_Y(q^X_{\alpha_L}), \\ P_H &= 1 - F_Y(q^X_{\alpha_H}). \end{aligned} \tag{3}$$

Notice that in this way $P_L + P_M + P_H = 1$. In the particular case in which both $q^X_{\alpha_L}$ and $q^X_{\alpha_L}$ do not belong to the support of Y, then $P_L = 0$, $P_M = 1$ and $P_H = 0$.

Remark to S3. As already stated, P_L is not – in general – the probability that Y is smaller than X, nor P_H is the probability that Y is larger than X. They are just the probability that Y is smaller than q_{α_L} or larger than q_{α_H}. A trivial counter example is the case in which $Y = X$. According to the Proposal we should have a 2.5% probability of X being smaller than itself and a 2.5% probabability to be larger than itself. ∎

2.3 Two general comments

A first comment. In general, the comparison between X and Y can be performed following two different approches:

1. focusing on the *difference* $Z = Y - X$ and computing the probability that Z is smaller (larger) than zero, *i.e.* is the probability that X is larger (smaller) than Y, $P[Z < 0] = P[Y < X]$ ($P[Z > 0] = P[Y > X]$);

2. considering X and Y *separately* and computing the probabilities that X and Y take values inside a given range $[\delta_a, \delta_b]$, $P[\delta_a \le X \le \delta_b]$ and $P[\delta_a \le Y \le \delta_b]$.

The first approach requires the knowledge of the *joint* distribution of X and Y. The reference probabilities cannot be obtained from the marginal distributions only.

On the contrary, the second approach, which is the one chosen by the legislator, is based on the marginal distributions.

It is important to stress that the events for which X is larger (smaller) than Y are not *mutually exclusive* with respect to the events for which X and Y have values inside the range $[\delta_a, \delta_b]$.

A second comment. The range $[\delta_a, \delta_b]$ is chosen by the legislator as $[q^X_{\alpha_L}, q^X_{\alpha_H}]$. This introduces an important asymmetry between X and Y. In fact, the common reader would expect that if the value of X is in line with the value of Y with a probability P_M, then Y would be in line with the value of X with the same probability, *i.e* that $P_M(X \to Y) = P_M(X \to Y)$.
Moreover, one would also expect the following relations to hold:

$$\begin{array}{rclcl} P[Y < X] & = & P_L(X \to Y) & = & P_H(Y \to X) = P[X > Y], \\ P[Y > X] & = & P_H(X \to Y) & = & P_L(Y \to X) = P[X < Y] \end{array} \quad (4)$$

Very unfortunately this is not the case for the definitions used by the Proposal. Imagine, for example two counterparties, one holding portfolio X and willing to reorganise it into portfolio Y, and the other holding portfolio Y and willing to reorganise it into portfolio X. Furthermore suppose that the support of Y is completely inside the range $[q^X_{\alpha_L}, q^X_{\alpha_H}]$. Then the first counterpart will set $P[Y < X] = P[Y > X] = 0$, while the second counterpart will find $P[X < Y] > 2.5\%$ and similarly $P[X > Y] > 2.5\%$.

3 Legal definition and correlations

The Proposal is based on the marginals only, since δ_a and δ_b are computed using $F_X(x)$, and the probabilities corresponding to the three scenarios are computed using $F_Y(y)$ only. In other words – according to the Proposal – the *correlation between X and Y is irrelevant*. However, the probability that Y is smaller than X is in general depending on the correlation between X and Y (the case $Y = X$ is a trivial example). Moreover, the values of the two financial portfolios are correlated through their common dependence to the evolution of the term structure of interest rates. In this section, we will give some further mathematical details on this point.

3.1 Legal definitions in the copula framework

For simplicity sake let's assume that $X \in \mathbb{R}$ and that $Y \in \mathbb{R}$ have both *continuous*[2] distributions $F_X(x)$ and $F_Y(y)$. In this case there exists and is unique

[2] Differently, the level of technicality should be increased. However the conclusions presented in this section would not change.

(the so-called Sklar theorem) a *copula function* $C : [0,1]^2 \to [0,1]$ such that:

$$F(x, y) = C\Big(F_X(x), F_Y(y)\Big) \tag{5}$$

In general, d-dimensional copula functions are defined in $[0,1]^d \to [0,1]$. They have two important properties:

1) $C(u_1, \dots u_d)$ is increasing in each component u_i;
2) $C(1, 1, \dots u_k, \dots 1, 1) = u_k$ for all $k = 1, \dots d$.

When the joint distribution function is continuous and the marginals are *strictly* increasing, the joint probability density $f_{XY}(x, y)$ can be expressed unsing the marginal proabability density functions $f_X(x)$ and $f_Y(y)$ and the density of the copula function $c(u, v)$, that is

$$f(x, y) = f_X(x)\, f_Y(y)\, c(u_x, u_y)$$

where

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v},$$

$u_x = F_X(x)$ and $u_y = F_Y(y)$.

Remark The copula corresponding to indipendent variable is $C(u, v) = u\, v$, and its density is $c(u, v) = 1$. ∎

Copula are useful because they allow to express joint distributions as functions of their marginals. Moreover, it is possible to show that the value of a copula is bounded both from above and below. In particular, for any d-dimensional copula

$$\max\Big\{\sum_{i=1}^{d} u_i + 1 - d, 0\Big\} \le C(\mathbf{u}) \le \min\{u_1, \dots u_d\}$$

In the literature the upper bound $\min\{u_1, \dots u_d\}$ is called the *comonotonicity* copula and corresponds to the case of "perfectly positively dependent" variables. The lower bound (in the two dimensional case) is called the *countermonotonicity* copula and corresponds to "perefctly negative dependent" variables. In the copula framework, the legal definitions can be expressed as

$$\begin{aligned} P[Y < X] \quad &:= \quad F_Y(\delta_a) = C\Big(1, \delta_a\Big) \\ P[\delta_a \le Y - X \le \delta_b] \quad &:= \quad F_Y(\delta_b) - F_Y(\delta_a) = C\Big(1, \delta_b\Big) - C\Big(1, \delta_a\Big) \end{aligned}$$

where

$$\delta_a = F_X^{-1}(\alpha), \delta_b = F_X^{-1}(1 - \alpha), \quad \text{and} \quad \alpha = 2.5\%.$$

Eq. (5) shows clearly that there are as many joint distribution functions as copula functions, all having the same marginals $F_X(x)$ and $F_Y(y)$.

3.2 The general expressions

For simplicity sake, let's assume that $X \in [a_X, b_X] \subseteq \mathbb{R}$ and $Y \in [a_Y, b_Y] \subseteq \mathbb{R}$ are continuous random variables with *joint* probability density function $f_{XY}(x, y)$.

The comparison between X e Y. The comparison between X and Y can be performed introducing a new random variable $Z \in [a_X, b_X] \subseteq \mathbb{R}$, given by the difference between Y and X. Using Z it is possible to define:

1. the probability that X is larger than Y, $P[Y - X < 0]$;
2. the probability that X and Y diffear at most by an amount included in the interval $[\delta_a, \delta_b]$, $P[\delta_a \leq Y - X \leq \delta_b]$, where δ_a and δ_b are exogenously given.

In the first case there are two reference scenarios $(X > Y, X \leq Y)$, while in the second case there are three reference scenarios $(Z < \delta_a, Z \in [\delta_a, \delta_b], Z > \delta_b)$

Proposition 1. The probability that X is larger than Y is

$$P[Y < X] = P[Y - X < 0] = \int_{a_Z}^{0} \Big[\int_{a_X}^{b_X} f_{XY}(w, z + w)dw\Big]dz \tag{6}$$

Proposition 2.The probability that $Y - X$ in included in $[\delta_a, \delta_b]$ is

$$P[\delta_a \leq Y - X \leq \delta_b] = \int_{\delta_a}^{\delta_b} \Big[\int_{a_X}^{b_X} f_{XY}(w, z + w)dw\Big]dz \tag{7}$$

Example. For a two-dimensional Normal distribution one has

$$N_2(x, y) = \frac{e^{-\frac{1}{2(1-\rho^2)}\Big[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)}{\sigma_X}\frac{(y-\mu_Y)}{\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\Big]}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}},$$

so that:

$$h(z) = \int_{-\infty}^{\infty} N_2(x, z + x)dx = \frac{e^{-\frac{1}{2}\frac{(z + \mu_X - \mu_Y)^2}{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}}}{\sqrt{2\pi}\sqrt{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}}. \tag{8}$$

The probailities $P[Y \leq X]$ and $P[\delta_a \leq Y - X \leq \delta_b]$ are respectively

$$P[Y \leq X] = \int_{-\infty}^{0} h(z)dz = \Phi\left(\frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}}\right) \tag{9}$$

and

$$P[\delta_a \le Y - X \le \delta_b] = \int_{\delta_a}^{\delta_b} h(z)dz =$$

$$= \Phi\left(\frac{\delta_b + \mu_X - \mu_Y}{\sqrt{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}}\right) - \Phi\left(\frac{\delta_a + \mu_X - \mu_Y}{\sqrt{\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2}}\right) \tag{10}$$

$\Phi(x)$ being the distribution function of a standard Normal distribution.

Numerically, let's examine the case of two independent standard Normal distributions. In this case:

$$\begin{gathered} \mu_X = \mu_Y = 0, \\ \sigma_X = \sigma_Y = 1, \\ \rho = 0, \\ \delta_a = \Phi^{-1}(0.025) = -1.96, \\ \delta_b = \Phi^{-1}(0.975) = 1.96 \end{gathered}$$

Moreover $P[Y \le X] = 0.5$ and $P[\delta_a \le Y - X \le \delta_b] = 0.8342$. Differently, *ex-lege*, one has $P_L = 2.5\%$, $P_H = 2.5\%$ and

$$P_M = \int_{\Phi^{-1}(0.025)}^{\Phi^{-1}(0.975)} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} = 0.95.$$

Notice that redefining $\delta_a = \phi^{-1}(0.0028) = -2.77$ and $\delta_b = \phi^{-1}(1 - 0.0028) = 2.77$ one would find exactly $P[\delta_a \le Y - X \le \delta_b] = 0.95$. ∎

The above example shows that although in general

$$P[Y < X] \ne F_Y(q^x_{\alpha_L}), \tag{11}$$

however there might exist joint distribution functions $F(X, Y)$ for which there exist $\hat{q}^x_{\alpha_L} \in \mathbb{R}$ such that

$$P[Y < X] = \int_{a_Z}^{0} \Big[\int_{a_X}^{b_X} f_{XY}(w, z + w)dw\Big]dz = F_Y(\hat{q}^x_{\alpha_L}). \tag{12}$$

It is therefore interesting to investigate the class of distributions for which the distribution of $Z = Y - X$ and the distribution of Y are identical up to a scale factor and a translation, so that eq. (12) can hold. We should refer to distributions that satisfy eq. (12) as "*equivalent distributions*". In the following paragraph we investigate this class of distributions.

4 The class of "*equivalent distributions*"

In the following paragraphs we address the problem of finding the class of bivariate distributions $F(X, Y)$, with marginal distributions $F_x(x)$ and $F_y(y)$, for which the linear combination $Z = aX + bY$ is F_y-distributed, up to a scale factor and a translation. For simplicity, we have restricted our analysis to the case of continuous random variables.

We believe that a proper classification of these distributions should distinguish them according to:

1. whether X and Y are independent or not, with the following sub-cases:
   (a) X and Y are independent;
   (b) X and Y are sub-independent;
   (c) X and Y are dependent, but not sub-independent;
2. whether the two marginal distributions are different or not.

In the following we will discuss the case of identical marginal distributions for all three cases of independence, sub-independence and dependent variables, giving some examples. The examples are taken from the literature and, with the exception of the case of sub-independence, are rather classic.
On the other hand – to the best of our knowledge – the case of different marginals has never been addressed. We have left this subject for future work.

4.1 Independent variables

When X and Y are independent their sum is given by the convolution and their difference by the cross correlation. The class of distributions closed under convolution is the class of *stable distributions*.

4.1.1 Stable distributions

The theory of univariate stable distributions was developed in the 1920s and 1930s by P. Lèvy and A. Khinchine. Modern reference textbooks are Zolotarev [10], Samorodnitsky and Taqqu [9] and Nolan [7]. References for financial applications are [6] and [5].

Definition. A non-degenerate random variable X is said to have a stable distribution if and only if for any $n > 1$ there exists two constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that

$$X_1 + X_2 + \ldots X_n \sim c_n\, X + d_n$$

where X_1, X_2, $\dots X_n$ are i.i.d. copies of X. Moreover, X is said to be *strictly stable* if $d_n = 0$. It can be shown [3, Section VI.1.1] that necessarily $c_n = n^{1/\alpha}$ with $\alpha \in (0, 2]$. The number α is called *index of stability* or *characteristic exponent*.

Remark. Notice that, differently from the hypotheses of the central limit theorem, there are no requirements on the moments of X. ■

Parametrization. Distributions of stable random variables are defined by mean of four parameters: $X \sim S(\alpha, \beta, \gamma, \delta)$, where $\alpha \in (0, 2]$ is the index of stability, $\beta \in [-1, 1]$ is a *skewness* parameter, $\gamma > 0$ is a *scale* parameter and $\delta \in \mathbb{R}$ is a *location* parameter. Apart from few exceptions there is no analytic expression available for $S(\alpha, \beta, \gamma, \delta)$; on the contrary its characteristic function, $\varphi(u)$, is known:

$$\varphi(u) = E\big[e^{iuX}\big] = \begin{cases} \exp\Big[-\gamma^\alpha |u|^\alpha [1 - i\beta(\tan\frac{\pi\alpha}{2})\mathrm{sign}(u)] + i\delta u\Big] & \alpha \neq 1 \\ \\ \exp\Big[-\gamma|u|[1 - i\beta\frac{2}{\pi}\mathrm{sign}(u)\log|u|] + i\delta u\Big] & \alpha = 1 \end{cases}$$

In the literature there exist several parametrizations for $\varphi(u)$; the one given above is the so-called Nolan "1" parametrization (see [7] for details).
Although the density of stable distributions is not known explicitely, the most relevant properties are well known.

Support. The support of stable distributions is:

$$\text{Support}\, S(\alpha, \beta, \gamma, \delta) = \begin{cases} [\delta, \infty) & \alpha < 1, \beta = 1; \\ (-\infty, \delta] & \alpha < 1, \beta = -1; \\ (-\infty, \infty) & \text{any other case.} \end{cases}$$

Density. All stable distributions are continuous with infinitely differentiable densities. Moreover, they are unimodal, although there is no known formula for the location of the mode. At present, there are only three cases where it is possible to write down closed form expressions for the density, which correspond to the following distributions:

1. Gaussian: $S(2, 0, \gamma, \delta) \sim \mathrm{N}(\delta, 2\gamma^2)$;
2. Cauchy: $S(1, 0, \gamma, \delta) \sim \mathrm{Cauchy}(\delta, \gamma) = \dfrac{1}{\pi}\dfrac{\gamma}{\gamma^2 + (x - \delta)^2}$;
3. Lévy: $S(\frac{1}{2}, 1, \gamma, \delta) \sim \text{Lévy}(\delta, \gamma) = \sqrt{\dfrac{\gamma}{2\pi}}\dfrac{1}{(x - \delta)^{3/2}} e^{-\dfrac{\gamma}{2(x - \delta)}}$.

Moments. Stable distributions admit moments only in special cases. The mean is δ when $\alpha > 1$, otherwise $E[X]$ is not finite. Similarly, the variance is finite only for $\alpha = 2$.

Remark. Notice that stable distributions with finite expected value ($\alpha > 1$) have support $(-\infty, \infty)$. ■

Linear combinations. A linear combination of stable random variables with the same index of stability is still a stable random variable. More precisely, let $X_j \sim S(\alpha, \beta_j, \gamma_j, \delta_j)$ be independent stable random variables, and ω_j be arbitrary constants, with $j = 1, \dots n \in \mathbb{N}$, then

$$Z = \sum_{j=1}^{n} \omega_j X_j \sim S(\alpha, \beta, \gamma, \delta)$$

where

$$\begin{aligned} \gamma^\alpha &= \textstyle\sum_{j=1}^{n} |\omega_j \gamma_j|^\alpha \\ \beta &= \frac{\sum_{j=1}^{n} \beta_j \mathrm{sign}(\omega_j) |\omega_j \gamma_j|^\alpha}{\gamma^\alpha} \\ \delta &= \begin{cases} \sum_{j=1}^{n} \omega_j \delta_j & \alpha \neq 1 \\ \sum_{j=1}^{n} \omega_j \delta_j - \frac{2}{\pi} \sum_{j=1}^{n} \beta_j \omega_j \gamma_j log|\omega_j| & \alpha = 1 \end{cases} \end{aligned}$$

Remark. Notice that since γ is a scale parameter and δ is a location parameter, they do not change the *shape* of the distribution. Differently, changes in the asymmetry parameter β cannot be absorbed by a translation and a rescaling of Z. For example, when considering the sum of two Gaussian distributed random variables, one has

$$X_j \sim N(\mu_j, \sigma_j) = S(2, 0, \gamma_j, \delta_j) \quad j = 1, 2,$$

and

$$Z = X_1 + X_2 \sim S\left(2, 0, \sqrt{\gamma_1^2 + \gamma_2^2}, \delta_1 + \delta_2\right) = S\left(2, 0, \frac{1}{2}\sqrt{\sigma_1^2 + \sigma_2^2}, \mu_1 + \mu_2\right)$$

from which it is clear that Z is also Gaussian distributed

$$Z \sim S(2, 0, \sigma_s, \mu_s)$$

albeit with parameters $\sigma_s^2 = \sigma_1^2 + \sigma_2^2$ e $\mu_s = \mu_1 + \mu_2$. ■

A case of interest. Let X_1 and X_2 be independent stable variables with the same index of stability α. Moreover, let's assume that $\alpha > 1$ so that both admit finite expected values. The distribution of the difference $X_1 - X_2$ is

$$X_1 - X_2 \sim S\left(\alpha, \frac{\beta_1\gamma_1^\alpha - \beta_2\gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha}, (\gamma_1^\alpha + \gamma_2^\alpha)^{1/\alpha}, \delta_1 - \delta_2\right)$$

in order to $X_1 - X_2 \sim X_2$ up to a translation and a change of scale, the following condition should hold

$$\frac{\beta_1\gamma_1^\alpha - \beta_2\gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha} = \beta_2, \qquad\qquad i.e.$$

$$\begin{cases} \gamma_1 = \gamma_2\left[\dfrac{2\beta_2}{\beta_1 - \beta_2}\right]^{1/\alpha} & \text{if} \quad \beta_1 \neq \beta_2 \\ \text{any} \quad \gamma_1, \gamma_2 & \text{if} \quad \beta_1 = \beta_2 = 0 \end{cases}$$

Differently, in order to have $X_1 - X_2 \sim X_1$ up to a translation and a change of scale, it is necessary to have $\beta_1 = \beta_2 = 0$.

Therefore, in order to have $Y - X \sim Y$, at least up to a scale and a position factor, we have to assume that $\beta_1 = \beta_2 = 0$. Moreover, notice that if $\mathbf{E}[X] = \mathbf{E}[Y]$, then $\delta_1 = \delta_2 = \delta$, and $\mathbf{E}[Y - X] = 0$.

Final remark concerning the case of independence. For any couple of indipendent α-stable random variables (X_1, X_2) distributed as $\Big(S(\alpha, 0, \gamma_1, \delta_1), S(\alpha, 0, \gamma_2, \delta_2)\Big)$, with $\alpha \in (1, 2]$, the difference between the first and the second variable is distributed as the minuend, up to a scale factor and a location offset. The case of two normal distributions corresponds to $\alpha = 2$.

The support of the distributions is $(-\infty, \infty)$; therefore they cannot be used to describe the value of a bounded quantity. ■

4.2 Sub-independent variables

Let $\mathbf{X} = (X_1, X_2, \ldots X_d) \in \mathbb{R}^d$ be a random vector with probability distribution function $F(\mathbf{x})$ and characteristic function $\varphi(\mathbf{t})$. Components of $\mathbf{X}$ are said to be sub-independent when

$$\varphi(\mathbf{t}) = \prod_{i=1}^{d} \varphi_i(t), \quad \forall\, \mathbf{t} = \{t, t, \ldots t\} \in \mathbb{R}^d, \tag{13}$$

where $\varphi_i(t)$ is the characteristic function of X_i. Therefore the distribution of the sum $S = \sum_{i=1}^{d} X_i$ is the the same one would have obtained in case of independence. Thus, if the X_i are α-stable random variables, then S is also an α-stable random variable. Moreover, the distribution of the difference $X_1 - X_2$ is the same as the distribution of the sum $X_1 + X_2$ when X_2 is symmetric, *i.e.* when $f_2(x) = f_2(-x)$.

In [4, §40, pag. 245] the case $d = 2$ is used to give an example of dependent uncorrelated random variables; as discussed there two r.v. X and Y with pdf's $f_x(x)$ and $f_y(y)$, and with joint probability density function

$$f_{xy}(x,y) = f_x(x)f_y(y) + \phi(x)\psi(y) - \psi(y)\phi(x), \tag{14}$$

are uncorrelated when

1. $\phi(x) \neq \psi(x)$;
2. $\int_{-\infty}^{\infty} \phi(x)dx = \int_{-\infty}^{\infty} \psi(x)dx = 0$;
3. $f_{xy}(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$.

In fact, the notion of correlation requires a finite second moment, and does not apply for stable distributions having $\alpha < 2$. However, the notion of sub-independence is always well defined and allows to construct useful examples.

Example. The special case of bivariate uncorrelated random variables with symmetric marginals has been recently been discussed in [2]. Using the same notation, we let

$$f_\beta(x_1,x_2) = f_1(x_1)f_2(x_2)\Big[1 + \beta q(x_1,x_2)\Big], \tag{15}$$

where $f_i(x_i)$, $i = 1,2$ are the marginal pdf's, $\beta \in \mathbb{R}$, $q(x_1,x_2)$ is any measurable bounded function on $\mathbb{R}^2$ with bound $|q(x_1,x_2)|\leq B = |\beta|^{-1}$, satisfying

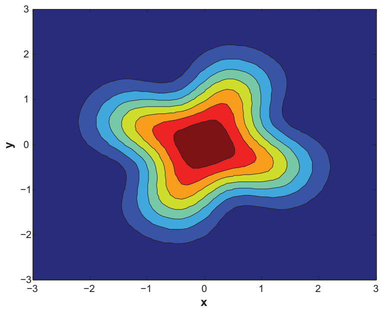
$$-q(x_1,x_2) = q(x_2,x_1) = q(-x_1,x_2) = q(x_1,-x_2). \tag{16}$$

Moreover, let both x_1 and x_2 be standard normal random variables with pdf's $f_i(x) = N(x;0,1)$ $(i = 1,2)$ and $q(x_1,x_2) = x_1\,x_2(x_1^2 - x_2^2)e^{-\frac{1}{2}(x_1^2+x_2^2)}$, so that

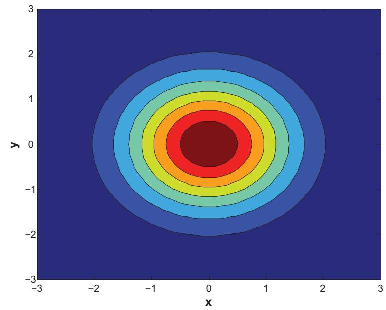
$$f_\beta(x_1,x_2) = \frac{e^{-\frac{1}{2}(x_1^2+x_2^2)}}{2\pi}\Big[1 + \beta\,x_1\,x_2\,(x_1^2 - x_2^2)e^{-\frac{1}{2}(x_1^2+x_2^2)}\Big],\ \beta \in \Big[0, \frac{e^2}{4}\Big], \tag{17}$$

where the bound on β can be easily derived by casting $f_\beta(x_1,x_2)$ in polar coordinates. The corresponding joint distribution function and copula are

$$F_\beta(x,y) = F_1(x_1)\,F_2(x_2) + \beta\frac{(x_1^2 - x_2^2)e^{-(x_1^2+x_2^2)}}{16\pi^2} \tag{18}$$



(a) dependent uncorrelated.



(b) independent.

Figure 1: Contour plot of the joint probability density function of two bivariate distributions with standard normal marginals (see text for details).

$$C_\beta(u,v) = u\,v + \beta \frac{\left(F^{-1}(u)^2 - F^{-1}(v)^2\right)e^{-\left(F^{-1}(u)^2+F^{-1}(v)^2\right)}}{16\pi^2} \tag{19}$$

In this case, the distribution of $Z = X_1 - X_2$ is normal with standard deviation $\sqrt{2}$, as it would have been obtained with from a bivariate standard normal with zero correlation. In fact:

$$f(z) = \int_{-\infty}^{\infty} f_\beta(x, z+x)dx = \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{z^2}{2}} = N(z; 0; \sqrt{2}) \tag{20}$$

The contour plot of the joint pdf's is shown in Fig. 4.2, respectively for the sub-independent case with $\beta = 0.5$ and the case of independence.

4.3 Dependent variables

The case of dependent variables is much more complex. We will examine the well-established case of mixtures of normal distributions and briefly discuss the case of multivariate stable distributions.

4.3.1 Normal mixture distributions

Normal mixture distributions are a generalisation of the multivariate normal distribution. Similarly to the multivariate normal distribution they are closed under linear transformations. In practice, this mean that all the margins share the same distribution F and that any linear combination of the margins is also F-distributed. The marginal distribution F depends on the choiche of the distribution H of the random mixing variable W, as shown in the next paragraphs.

Normal variance mixture

The random vector $\mathbf{X} = (X_1, \dots X_d) \in \mathbb{R}^d$ is said to have a normal variance mixture distribution when

$$\mathbf{X} = \boldsymbol{\mu} + W^{1/2} A \mathbf{Z} \tag{21}$$

where $\mathbf{Z} \sim N_p(\mathbf{0}, I_p)$, $W \geq 0$ is a non-negative, scalar-valued random variable independent of $\mathbf{Z}$, and $A \in \mathbf{R}^{d \times p}$ and $\boldsymbol{\mu} \in \mathbf{R}^d$ are a matrix and a vector of constants, respectively.
The mixture is such that, when conditioning on W, $\mathbf{X}$ is normally distributed, *i.e.*

$$(\mathbf{X}|W = w) \sim N_d(\boldsymbol{\mu}, w\Sigma), \quad \text{where} \quad \Sigma = AA^T. \tag{22}$$

Assuming that Σ is positive definite and that the distribution of W has no point mass at zero, the joint density of $\mathbf{X}$ is given by

$$f(\mathbf{x}) = \int f(\mathbf{x}|w) dH(w) = \int \frac{w^{-d/2}}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\dfrac{(\mathbf{x}-\boldsymbol{\mu})\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})^T}{2w}} dH(w) \tag{23}$$

where $H(w)$ is the distribution function of W.

Eq. (23) shows that $f(\mathbf{x})$ depends on $\mathbf{x}$ only through the quadratic form $(\mathbf{x}-\boldsymbol{\mu})\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})^T$, thus $f(\mathbf{x})$ belongs to the class of the so-called *elliptical distributions*. Moreover, provided W has a finite expectation, the first two moments of $\mathbf{X}$ are

$$\begin{aligned} E[\mathbf{X}] &= E[\boldsymbol{\mu} + \sqrt{W} A\mathbf{Z}] = \boldsymbol{\mu} + E[\sqrt{W}]A\,E[\mathbf{Z}] = \boldsymbol{\mu} \\ \operatorname{cov}[\mathbf{X}] &= E[(\sqrt{W}A\mathbf{Z})\,(\sqrt{W}A\mathbf{Z})^T] = E[W]AE[\mathbf{Z}\mathbf{Z}^T]A^T = E[W]\Sigma \end{aligned} \tag{24}$$

Despite the fact that the covariance between any two components of $\mathbf{X}$, $\operatorname{cov}(X_i, X_j)$ $(i \neq j)$ is null when $\Sigma = I_d$, this is not sufficient to make X_i and X_j independent. In fact, independence also requires W to be constant and corresponds to the degenerate case of a multinormal distribution with null covariance.

An important property of normal mixture distributions is that they are closed under linear transformations, *i.e.* if $\mathbf{X} \sim f(\mathbf{x}) = M_d(\boldsymbol{\mu}, \Sigma, \hat{H})$, where $\hat{H}$ is the Laplace-Stieltjes transform of the distribution function $H(w)$ of W, and $\mathbf{Y} = B\mathbf{X} + \mathbf{b}$, with $B \in \mathbb{R}^{k \times d}$ and $\boldsymbol{b} \in \mathbf{R}^k$ a matrix and a vector of constants, then

$$\mathbf{Y} \sim M_k(B\boldsymbol{\mu} + \boldsymbol{b}, B\Sigma B^T, \hat{H}). \tag{25}$$

In other words, any linear combination of the components of $\mathbf{X}$ is distributed like the components themselves, that are all distributed in the same way. Notice, however, that $M_d(\boldsymbol{\mu}, \Sigma, \hat{H})$ is *not* a stable distribution.

Special cases of normal variance mixture distributions are the multivariate t distribution with ν degrees of freedom, that is obtained by choosing $H(w)$ as an inverse Gamma distribution, $W \sim Ig(-\nu/2, -\nu/2)$, and the generalised hyperbolic distribution that is obtained by chosing $H(w)$ as a generalised inverse Gaussian (GIG) distribution, $W \sim N^{-}(\lambda, \chi, \psi)$ (see [5, §3.2, pag. 75] for details).

Example. Let's considerate a multivariate t distribution. Solving the integral in eq. (23) the joint pdf is given by

$$f_d(\mathbf{t}; \boldsymbol{\mu}, \Sigma) = \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{d/2}|\Sigma|^{1/2}} \left[1 + \frac{(\mathbf{t}-\boldsymbol{\mu})\Sigma^{-1}(\mathbf{t}-\boldsymbol{\mu})^T}{\nu}\right]^{-\left(\frac{\nu+d}{2}\right)} \tag{26}$$

The marginal distribution of t_1 – a t distribution – can be easily computed by choosing $B = \begin{pmatrix} 1\, 0 \, \ldots 0 \end{pmatrix}$,

$$f(t_1) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{1/2}} \left[1 + \frac{(t_1-\mu_1)^2}{\nu}\right]^{-\left(\frac{\nu+1}{2}\right)}. \tag{27}$$

Similarly, the distribution of the weighted sum $S = \omega_1 t_1 + \omega_2 t_2$ is found by choosing $B = \begin{pmatrix} \omega_1\, \omega_2\, 0 \, \ldots 0 \end{pmatrix}$

$$f(S) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{1/2}\omega_S^{1/2}} \left[1 + \frac{(S-\mu_S)^2}{\omega_S \nu}\right]^{-\left(\frac{\nu+1}{2}\right)}. \tag{28}$$

where $\mu_s = \omega_1\mu_1 + \omega_2\mu_2$ and $\omega_S = B\Sigma B^T = (\omega_1^2+\omega_2^2)(1+\rho)$. It is evident that $S/\omega_S^{1/2} \sim t_1$.

Finally, notice that when when t_1 and t_2 are two independent t-distributed random variables with ν degrees of freedom each, the distribution of the sum $S_I = t_1 + t_2$ is *not* a t distribution (see [8] for details) and its not found as the limiting case of the distribution of S for $\omega_1 = \omega_2 = 1$ and $\rho \to 0$.

Normal variance-mean distributions

Normal variance-mean distributions are extensions of normal variance distributions. The random vector $\mathbf{X} = \{X_1, \ldots X_d\} \in \mathbb{R}^d$ is said to have a normal variance mixture distribution when

$$\mathbf{X} = \boldsymbol{\mu} + W\boldsymbol{\gamma} + W^{1/2}A\mathbf{Z} \tag{29}$$

where $\mathbf{Z} \sim N_p(\mathbf{0}, I_p)$, $W \geq 0$ is a non-negative, scalar-valued random variable independent of $\mathbf{Z}$, and $A \in \mathbf{R}^{d\times p}$ and $\boldsymbol{\mu}, \boldsymbol{\gamma} \in \mathbf{R}^d$ are a matrix and two vectors of constants, respectively.

A special case is given by the full generalized hyperbolic distribution [1] that is obtained choosing $H(w)$ as a generalised inverse Gaussian (GIG) distribution, $W \sim N^{-}(\lambda, \chi, \psi)$. This distribution has a density

$$f(\mathbf{x}) = N_c \frac{K_{\lambda-d/2}\left(\sqrt{\left(\chi + (\mathbf{x}-\boldsymbol{\mu})\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})^T\right)\left(\psi + \boldsymbol{\gamma}\Sigma^{-1}\boldsymbol{\gamma}^T\right)}\right) e^{(\mathbf{x}-\boldsymbol{\mu})\Sigma^{-1}\boldsymbol{\gamma}^T}}{\left[\left(\chi + (\mathbf{x}-\boldsymbol{\mu})\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})^T\right)\left(\psi + \boldsymbol{\gamma}\Sigma^{-1}\boldsymbol{\gamma}^T\right)\right]^{d/2-\lambda}}, \tag{30}$$

where the normalization constant N_c is given by

$$N_c = \frac{(\chi\psi)^{-\lambda/2}\psi^{\lambda}\left(\psi + \boldsymbol{\gamma}\Sigma^{-1}\boldsymbol{\gamma}^T\right)^{d/2-\lambda}}{(2\pi)^{d/2}|\Sigma|^{1/2}K_{\lambda}(\sqrt{\chi\psi})}, \tag{31}$$

and $K_\lambda(x)$ is the modified Bessel function of the third kind. Notice that $f(\mathbf{x})$ is *elliptical* only when $\boldsymbol{\gamma} = \mathbf{0}$.

The full generalized hyperbolic distribution is also closed under linear operations, *i.e.* if $\mathbf{X} \sim f(\mathbf{x}) = GH_d(\lambda, \chi, \psi, \boldsymbol{\mu}, \Sigma, \boldsymbol{\gamma})$ and $\mathbf{Y} = B\mathbf{X}+\mathbf{b}$, with $B \in \mathbb{R}^{k\times d}$ and $\boldsymbol{b} \in \mathbf{R}^k$ a matrix and a vector of constants, then

$$\mathbf{Y} \sim GH_k(\lambda, \chi, \psi, B\boldsymbol{\mu} + \boldsymbol{b}, B\Sigma B^T, B\boldsymbol{\gamma}). \tag{32}$$

This property implies that the margins of $\mathbf{X}$ are all distributed as univariate generalized hyperbolic distribution $X_i \sim GH_1(\lambda, \chi, \psi, \mu_i, \Sigma_{ii}, \gamma_i)$, and that any linear combinations of margins is still distributed as an univariate generalized hyperbolic distribution.

The generalised hyperbolic distribution is widely used since it is possible to construct a Lévy process for which the value of the increment of the process over a fixed time interval has a given generalised hyperbolic distribution. This is possible since the distribution is a so-called infinitely divisible distribution, a property inherited from the GIG mixing distribution of W.

4.3.2 Multivariate stable random variables

We have seen that normal mixtures are closed under linear combinations. This property is essentially inherited from the multivariate Normal. It is therefore natural to ask what happens if we generalise to the case of a multivariare α-stable distribution instead of a multivariate Normal.

Let's first recall that a vector $\mathbf{X} \in \mathbb{R}^d$ is called *stable* if and only if for any $n > 1$ there is a constant $\alpha \in (0, 2]$ and a vector $D_n \in \mathbb{R}^d$ such that

$$\mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \ldots \mathbf{X}^{(n)} = n^{1/\alpha}\,\mathbf{X} + \mathbf{D}_n \tag{33}$$

where $\mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \ldots \mathbf{X}^{(n)}$ are *independent copies* of $\mathbf{X}$.

Remark. As in the case of the multivariate Normal distribution – in general – the components of $\mathbf{X}$ *are not independent*, although it is possible to construct multivariate stable random vectors with independent components. ∎

Proposition. If $\mathbf{X}$ is an α-stable random vector and $\mathbf{b} \in \mathbb{R}^d$ is a vector of constants, then all linear combinations

$$Y = (\mathbf{b}, \mathbf{X}) = \sum_{i=1}^{d} b_i X_i$$

are α-stable random variables. In general, the converse *is not* true. It is true only if all linear combinations are *strictly* stable, or if $\alpha \geq 1$. Thus, as very well-known, in the Gaussian case $\alpha = 2$ the converse is true.

Unfortunately, properties of multivariate stable vectors and of their mixtures are not much studied, due to the lack of analytic expressions. When $\mathbf{X} \sim \mathbf{G}$ where $\mathbf{G} = N(\mathbf{0}, \Sigma)$ is a zero mean multivariate normal distribution, the corresponding normal variance mixtures are knwon under the name of *sub-Gaussian* distributions.

4.3.3 Sub-Gaussian distributions

Normal mixtures where the mixing variable W is also an α-stable variable are called *sub-Gaussian*. Remember that in order to have support $[0, \infty)$ a stable r.v. should have $\alpha \in (0, 1)$, $\beta = 1$ and $\delta = 0$. In this case, the Laplace transform $\hat{H}(t)$ of $W \sim S(\alpha, 1, \gamma, 0)$ is given by

$$\hat{H}(t) = \mathbf{E}\Big[e^{-t\,W}\Big] = e^{-\dfrac{t^\alpha\,\gamma^\alpha}{\cos(\frac{\pi\,\alpha}{2})}} \tag{34}$$

Since for any stable distribution $\alpha \leq 2$, it is customary to use $\alpha/2$ instead of α for totally right skewed stable distributions.

Proposition. Let $\mathbf{G} \sim N(\mathbf{0}, \Sigma)$ be a multivariate normal random vector with zero mean and covariance matrix Σ, and let $W \sim S\Big(\dfrac{\alpha}{2}, 1, \gamma, 0\Big)$ be an independent univariate $(\alpha/2)$-stable random variable, then

$$\mathbf{X} = W^{1/2} G$$

is an α-stable elliptically contoured r.v. with joint characteristic function

$$\varphi(\mathbf{t}) = \mathbf{E}\Bigg[e^{\,i\sum_{k=1}^{d} t_k\,X_k}\Bigg] = e^{-\left(\frac{\gamma}{2}\right)^{\alpha/2}\,\sec\left(\frac{\pi\,\alpha}{4}\right)(\mathbf{t}^T\,\Sigma\,\mathbf{t})^{\alpha/2}}. \tag{35}$$

When $\gamma = \cos(\frac{\pi\alpha}{4})^{2/\alpha}$, W has Laplace transform $\hat{H}(t) = \exp\Big(-t^{\alpha/2}\Big)$, while $\mathbf{X}$ has characteristic function $\varphi(\mathbf{t}) = \exp\Big(-\frac{1}{2}(\mathbf{t}\,\Sigma\,\mathbf{t}^T)^{\alpha/2}\Big)$. In this case $\mathbf{X}$ is called a *sub-Gaussian* random vector.

Remark. Notice that for sub-Gaussian random vectors the distribution of the mixing variable W depends only upon α. ■

Example. The multivariate Cauchy distribution has characteristic function

$$\varphi(\mathbf{t}) = e^{-\dfrac{1}{2}(\mathbf{t}\,\Sigma\,\mathbf{t}^T)^{1/2} + i\boldsymbol{\mu}t^T}, \tag{36}$$

and density

$$f(\mathbf{x}) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}|\Sigma|^{1/2}} \frac{1}{\Big[1 + (\mathbf{x}-\boldsymbol{\mu})\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})^T\Big]^{(d+1)/2}}, \quad \mathbf{x} \in \mathbb{R}^d, \tag{37}$$

thus it is a so-called *shifted* sub-Gaussian distribution with shift $\boldsymbol{\mu}$, $\alpha = 1$ and $W \sim S(\frac{1}{2}, 1, \frac{1}{2}, 0)$, which is a Lévy$(0, \frac{1}{2})$ distribution. ■

Proposition. For a $\boldsymbol{\mu}$ shifted sub-Gaussian r.v. $\mathbf{X}$ the linear combination $Y = (\boldsymbol{b}, \mathbf{X})$ is an α-stable r.v. $S\Big(\alpha, \beta(\mathbf{b},\boldsymbol{\mu}), \gamma(\mathbf{b},\boldsymbol{\mu}), \delta(\mathbf{b},\boldsymbol{\mu})\Big)$ with

$$\beta(\mathbf{b},\boldsymbol{\mu}) = 0, \quad \gamma(\mathbf{b},\boldsymbol{\mu}) = (1/2)\Big(\mathbf{b}\Sigma\mathbf{b}^T\Big)^{\alpha/2}, \quad \delta(\mathbf{b},\boldsymbol{\mu}) = \boldsymbol{b}\boldsymbol{\mu}^T. \tag{38}$$

Remark. Notice that the support of Y is $\mathbb{R}$ since $\beta = 0$. Moreover, Y has finite mean when $\alpha > 1$, *i.e.* when the index of stability of W is larger that $1/2$. ■

5 Conclusions

We have investigated the mathematical framework underlying the "scenarios" approach for derivate transactions by Italian local Authorities, as defined in the Proposal issued on September, 2009 by the Italian Ministry of Economy and Finance.
If X is the random variable representing the value at maturity of the original liability ("*portafoglio finanziario iniziale*") and – similarly – Y is the r.v. representing the value of the liability after the subscription of the derivative by the local Authorithy ("*portafoglio strutturato*"), a critical issue is that the probabilities of the three scenarios defined by the Proposal are computed considering only the distribution functions of X and Y.
Consequently, the Proposal is unable to provide significant disclosure in those cases in which the support of Y is contained in the support of X (it could be the case when an interest rate swap is used to reorganise a floating rate debt in a fixed rate debt, that is a very common financial practice).
In addition – according to the Proposal – the correlation between X and Y is irrelevant.
In this context, we believe that a proper analysis should distinguish whether the marginal distributions of X and Y belong to the same family (are identical up to a scale factor and a translation) or not, and whether they are independent or not.
We have reviewed the case of identical marginal distributions for the following sub-cases: whether X and Y are independent; whether X and Y are sub-independent; whether X and Y are dependent (but not sub-independent), presenting several examples taken from the literature which are rather classic, with the exception of the case of sub-independence.
As discussed in par. 4 it is possible to construct several cases of bivariate random variables (X, Y) for which the difference $Y - X$ is distributed as Y up to a scale factor and a translation, which is the condition required to match the Proposal prescription for the computation of the scenario probabilities. The possibility of varying the dependence between X and Y allows to choose between a large number of joint distributions, ranging from the bivariate normal to the generalized hyperbolic distribution, or to unnamed non-analytical multivariate stable distributions. All the cases have in common the fact that the support of both marginals has to be $\mathbb{R}$ when requiring a finite mean of X and Y.
On the other hand – to the best of our knowledge – the case of different marginals has never been addressed. We have left this subject for future work.

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