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THE STRUCTURE OF COMPETITIVE EQUILIBRIUM WITH UNSECURED DEBT

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Abstract. I provide a complete characterization of equilibrium with risk of default in sequential economies under uncertainty. Default induces permanent exclusion from financial markets and not-too-tight solvency constraints prevent debt repudiation at equilibrium. The method of analysis relies on a recursive planning program along with the theory of monotone concave operators. The reputational mechanism is fragile, as it sustains constrained efficient as well as constrained inefficient equilibria. Constrained inefficient equilibria involve a progressive deterioration of reputation, inducing a collapse of financial markets with positive probability. Importantly, the only *ergodic* recursive equilibria (involving trade) are constrained efficient.

Keywords. Limited commitment; solvency constraints; competitive equilibrium; constrained efficiency; dynamic programming; monotone concave operator.

JEL Classification Numbers. D500, D520, D610, E440, G130.

1. Introduction

The purpose of this paper is to study the debt enforcement mechanism in competitive economies. Debtors might not deliver on their promises. Upon default, all assets are seized and the debtor is punished by a permanent exclusion from market participation, because of a complete loss of reputation (Eaton and Gersovitz [13], Kehoe and Levine [18, 19], Kocherlakota [22] and Alvarez and Jermann [4]). At equilibrium, default is prevented by endogenously determined debt limits, quantitative bounds specific to individuals and contingencies, which enforce the maximum expansion of risk-sharing subject to individual rationality of debt repayments (Alvarez and Jermann [4]). The enforcement mechanism is intrinsically fragile: as debt is not secured by any collateral, self-fulfilling expectations on the deterioration of solvency conditions might induce excess volatility of prices, poor volumes of trade and severe credit restrictions, with dramatic implications for social welfare due to residual uninsured risk. The extreme feature of this fragility is an immediate complete collapse of markets (Alvarez and Jermann [4, Proposition 4.3]).

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In spite of a common sense for multiplicity, no substantial effort has been devoted to a deeper understanding of the structure of competitive equilibrium under risk of default. For instance, along with constrained efficient risk-sharing, could competitive markets also sustain stationary equilibria with low volumes of trade? Could autonomous revisions of expectations induce a persistent excess volatility of prices? Could the economy benefit from policy interventions, or market regulations, preventing some equilibria with unsatisfactory social welfare? The absence of previous studies is partly due to analytical obstacles. Established characterizations are limited to simple economies with only two individuals (see, for instance, Ljungqvist and Sargent [28, Chapter 20]). Equilibrium dynamics are peculiar under the restrictive assumption of two individuals, mostly because of the so-called amnesia property (equilibrium retains no information of past dynamics whenever one of the two individuals is constrained). The nature of competitive equilibria in more general economies remains unexplored.

I here develop a recursive method permitting a complete characterization of competitive equilibrium under limited commitment. This method relies on the techniques of Abreu, Pearce and Stacchetti [1] along with an innovative utilization of the theory of monotone concave operators (Krasnosel'skiĭ [23]). Though the approach is analytically sophisticated and admittedly indirect, it bears the relevant advantage of uncovering the entire structure of the equilibrium set for general economies under minimal assumptions on fundamentals. Before presenting the positive theory of competitive equilibrium, I briefly describe the method, as this might be of autonomous interest and of potential broad applicability.

The approach moves from a basic observation: as no pecuniary externality affects the value of default (permanent interdiction from trades), the allocation of risk is socially optimal, subject to no default constraints, over any arbitrary finite horizon (Bloise, Reichlin and Tirelli [8]). This permits to study competitive equilibrium by means of a sort of recursive planning program, as in the previous literature on optimal contracts under limited commitment (most prominently, Kocherlakota [22], Phelan [32] and Thomas and Worrall [34]; see also Benhabib and Rustichini [9] for dynamic programming with incentive constraints). The theoretical construct unfolds as follows: At every contingency, an hypothetical benevolent planner distributes *current* consumptions and *continuation* utilities, subject to participation, or no default, constraints; consumptions fulfill material feasibility restrictions; continuation utilities are constrained by some feasible set; the planning program, which is conditional on given feasible sets for continuation utilities, yields readjusted feasible sets for continuation utilities; consistency obtains only when this revision process ceases. As matter of fact, this construction defines a sort of Bellman operator, as in canonical recursive methods adopted in macroeconomics (among others, Lucas and Stokey [27]). However, differently from the traditional applications, the planning program is not restricted to reflect only intrinsic uncertainty affecting fundamentals. Therefore, the domain of the operator is vast: feasible sets for utilities might be history dependent, varying over time and contingencies, possibly only because of autonomous revisions of expectations. This is necessary in order to recover the entire set of (potentially non-stationary) competitive equilibria.

Fixed points of the Bellman operator correspond to (contract curves of) competitive equilibria with risk of default. Furthermore, they completely exhaust the entire set of competitive equilibria. Therefore, an understanding of the structure of competitive equilibria requires to study the entire set of fixed points of the Bellman operator, a multiplicity varying from autarchy to constrained efficiency. Remarkably, the cost of abstraction is compensated by a high return in terms of a stringent characterization.

Though it fails the contraction property, the Bellman operator inherits a relevant feature of monotone concavity which considerably restricts the structure of fixed points. Monotonicity reflects the fact that more permissive feasible sets for continuation utilities result in a higher social value for the planning program. Concavity derives from the convexity of feasible sets, along with concavity of utilities; this property is not immediate, as it involves weighted sums of feasible sets for continuation utilities; importantly, strict concavity can be established under sufficiently permissive restrictions on fundamentals. Previous economic applications of monotone concave operators, to the best of my knowledge, are limited to the representation of recursive preferences (Marinacci and Montrucchio [31]) and to optimal growth (LeVan, Morhaim and Vailakis [25]); an elementary result for uniqueness is provided by Kennan [21]. I now present the major achievements of the analysis. For an intuitive grasp of the logic underlying results, it is helpful to heuristically under-represent the Bellman operator as an increasing concave map on the line admitting a minimal fixed point at the origin (autarchy) and a maximal fixed point in the positive cone (constrained efficiency).

First, the economy admits a unique (non-autarchic) Markov equilibrium on a minimal state space, consisting of the exogenous state affecting fundamentals and of welfare weights accounting for the distribution of welfare (wealth) across traders. Furthermore, this simple Markov equilibrium achieves constrained efficiency. Uniqueness is the counter-part of strict concavity of the Bellman operator: An increasing strictly concave map admits at most one fixed point in the positive cone. Importantly, this does not exclude the existence of (constrained inefficient) Markov equilibria on a larger state space, possibly including extrinsic, or sunspot, uncertainty.

Second, the economy admits a continuum of other non-stationary equilibria monotonically converging to the autarchy. In these equilibria, reputation progressively deteriorates, solvency restrictions become more severe and volumes of trade reduce over time up to a complete collapse of markets. This is a non-immediate implication of monotone concavity: Intuitively, the Bellman operator represents backward dynamics at a competitive equilibrium, as it maps continuation utilities (the future) into current consumptions (the present); an increasing strictly concave map admits an orbit connecting the extreme fixed points; the backward branch of this orbit corresponds to equilibria converging to the autarchy. This characterization is exhaustive in deterministic economies, as in economies of overlapping generations under gross-substitutability (Kehoe, Levine, Mas-Colell and Woodford [20]). In the admittedly more relevant case of uncertainty, equilibrium might exhibit a richer variety of dynamics, stochastically oscillating between the two extremes of constrained efficiency and autarchy. A simple instance of this is an equilibrium in which, according to the realization of a sunspot signal, the economy move either upwards, in the direction of constrained efficiency, or downwards, in the direction of autarchy.

Third, under some mild restrictions, any (non-autarchic) *ergodic* generalized Markov equilibrium is constrained efficient (and, hence, coincides with the unique minimal Markov equilibrium), irrespectively of the extension of the state space. When constrained efficiency fails, the economy will be with positive probability arbitrarily close to autarchy in the long-run. Indeed, iterations of an increasing strictly concave map converge to the maximal fixed point (constrained efficiency) when the initial condition is bounded away from the origin (autarchy). This in turn implies that the autarchy is an accumulation point of the equilibrium process. Therefore, a failure of constrained efficiency necessarily entails a collapse of markets with positive probability, thus violating ergodicity. Loosely interpreted, any disordered market behavior is a transitory condition involving a strong tendency to market collapse.

Instead of permanent exclusion from trade, variants of the equilibrium concept admit more lenient punishments upon default. Bulow and Rogoff [10] and Hellwig and Lorenzoni [15] (see also Dutta and Kapur [12]) provide a relevant instance of partial exclusion: Debt repudiation inhibits future borrowing, though lending remains unrestricted. Other examples include temporary exclusion for a limited number of periods or permanent exclusion with some probability (for instance, Azariadis and Kaas [6]). The method in this paper does not apply directly because of the pecuniary externality, as reallocations of risk induce readjustments in prices which affect the value of default (the reservation utility of a trader). To what extent the technique can be adapted, at least in economies with the gross-substitutability property, remains for future research.

The paper is organized as follows. In section 2, I describe the fundamentals of the economy. In section 3, I present the concept of competitive equilibrium with limited commitment and I refer to previous studies for suitable Welfare Theorems. In section 4, I introduce the planning program and I describe the basic properties of the Bellman operator. In section 5, I show that fixed points of the Bellman operator precisely exhaust the set of competitive equilibria. Finally, in section 6, I provide the characterization by exploiting monotone concavity of the Bellman operator. All proofs are collected in the appendix.

2. Fundamentals

2.1. **Time and uncertainty.** The economy extends over an infinite horizon,

$$\mathbb{T} = \{0, 1, 2, 3, \ldots, t, \ldots\},$$

subject to uncertainty. Uncertainty is represented by a probability space, $(\Omega, \mathcal{F}, \mu)$, and a filtration of σ-algebras,

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_t \subset \cdots \subset \mathcal{F}.$$

To simplify, and to avoid issues of integrability, it is assumed that $\mathcal{F}_0$ is the trivial σ-algebra and, for every t in $\mathbb{T}$, $\mathcal{F}_t$ is a σ-algebra generated by a *finite* partition of Ω. Given a state of nature ω in Ω, at every period t in $\mathbb{T}$, $\mu(\mathcal{F}_t(\omega)) > 0$, where $\mathcal{F}_t(\omega) = \cap\{E_t \in \mathcal{F}_t : \omega \in E_t\}$ represents the available information. In the equivalent event-tree representation of uncertainty, this corresponds to a date-event.

2.2. **Adapted processes.** Given any topological space V, endowed with its Borel σ-algebra, $L(V)$ denotes the space of all maps $f : \mathbb{T} \times \Omega \to V$ such that, for every t in $\mathbb{T}$, $f_t : \Omega \to V$ is $\mathcal{F}_t$-measurable, an element of $L_t(V)$. Whenever V is a Banach space, $L(V)$ decomposes as $\oplus_{t\in\mathbb{T}} L_t(V)$, where each component $L_t(V)$ is itself a

Banach space endowed with the norm

$$\|f_t\|_t = \sup_{\omega \in \Omega} \|f_t(\omega)\|_V .$$

A relevant linear subspace of $L(V)$ is $L^\infty(V)$, consisting of all elements f of $L(V)$ such that

$$\|f\|_\infty = \sup_{t \in \mathbb{T}} \|f_t\|_t \text{ is finite.}$$

The latter space, endowed with the supremum norm, inherits the Banach structure. However, the linear space $L(V)$ and, hence, its linear subspace $L^\infty(V)$ might also be endowed with the product topology generated by the metric

$$d(f', f'') = \sum_{t \in \mathbb{T}} \frac{1}{2^t} \frac{\|f'_t - f''_t\|_t}{1 + \|f'_t - f''_t\|_t}.$$

Under this topology, the sequence $(f^n)_{n \in \mathbb{N}}$ in $L(V)$ converges to f in $L(V)$ if and only if, for every t in $\mathbb{T}$, the sequence $(f^n_t)_{n \in \mathbb{N}}$ in $L_t(V)$ converges to f_t in $L_t(V)$. Finally, whenever V is a partially ordered space, $L(V)$ inherits the ordering. In particular, an element f in $L(V)$ is (weakly) *positive* if, at every t in $\mathbb{T}$, $f_t(\omega)$ is a (weakly) positive element of V for every ω in Ω. To avoid any misunderstanding, I remark the use of positive in a weak sense: For instance, an adapted process f in $L(\mathbb{R})$ is positive whenever, at every t in $\mathbb{T}$, $f_t(\omega) \geq 0$ for all ω in Ω; as usual, $f \geq 0$ denotes positivity and $f > 0$ non-null positivity.

2.3. **Preferences and endowments.** There is a finite set J of individuals. For every individual i in J, the *consumption space* X^i is the positive cone of the *commodity space* $L(\mathbb{R})$. For every individual i in J, the *endowment* e^i in X^i satisfies the following boundedness assumption.

Bounds of endowments. *There exists a sufficiently small* $1 > \epsilon > 0$ *satisfying, for every individual* i *in* J*, at every* t *in* $\mathbb{T}$,

$$\frac{1}{\epsilon} \geq e^i_t \geq \epsilon.$$

Every individual i in J is characterized by a bounded, smooth, strictly increasing and strictly concave per-period utility $u^i : \mathbb{R}_+ \to \mathbb{R}$ and a process π^i in $L(\mathbb{R})$ such that $\inf_{\omega \in \Omega} \pi^i_t(\omega) > 0$, for every t in $\mathbb{T}$, and $\mathbb{E} \sum_{t \in \mathbb{T}} \pi^i_t$ is finite. These induce preferences over the consumption space X^i by defining, at every t in $\mathbb{T}$,

$$U^i_t(x^i) = \frac{1}{\pi^i_t} \mathbb{E}_t \sum_{s \in \mathbb{T}} \pi^i_{t+s} \left(u^i\left(x^i_{t+s}\right) - u^i\left(e^i_{t+s}\right) \right).$$

Notice that inter-temporal utilities are defined as surpluses with respect to autarchy. The simple aggregator $V^i_t : L_t(\mathbb{R}) \times L_{t+1}(\mathbb{R}) \to L_t(\mathbb{R})$ is given by

$$V^i_t\left(x^i_t, w^i_{t+1}\right) = \left(u^i\left(x^i_t\right) - u^i\left(e^i_t\right)\right) + \frac{1}{\pi^i_t} \mathbb{E}_t \pi^i_{t+1} w^i_{t+1}.$$

Obviously, this construction is recursive, that is,

$$U^i_t(x^i) = V^i_t\left(x^i_t, U^i_{t+1}(x^i)\right).$$

The process π^i in $L(\mathbb{R})$ represents both varying impatience and subjective beliefs.

I impose *uniform impatience* over time and uncertainty (see, for instance, Levine and Zame [26, Assumption 5] or Santos and Woodford [33, Assumption 2]). It requires a bound on the marginal rate of substitution of current consumption for perpetual future consumption. The role of this hypothesis is technical: It ensures that utility surpluses are uniformly bounded over time and states of nature.

Uniform impatience condition. *There exists a sufficiently small* $1 > \eta > 0$ *satisfying, for every individual* i *in* J*, in every period* t *in* $\mathbb{T}$,

$$\pi_t^i \geq \eta \mathbb{E}_t \sum_{s \in \mathbb{T}} \pi_{t+s}^i.$$

Finally, preferences are restricted by a sort of weak form of Inada's condition. Its role is purely technical: It guarantees a uniform lower bound on consumption at every feasible allocation; at the same time, it avoids unbounded per-period utilities. Under acceptable hypotheses, any unbounded per-period utility could be modified so as to fulfill the boundary restriction without altering preferences over feasible allocations.

Boundary condition. *For every individual* i *in* J*, the per-period utility* $u^i : \mathbb{R}_+ \to \mathbb{R}$ *satisfies*

$$u^i(\epsilon) > \eta u^i(0) + (1 - \eta) u^i\left(\frac{\operatorname{card}(J)}{\epsilon}\right),$$

where $1 > \epsilon > 0$ *is given by the bounds on endowments and* $1 > \eta > 0$ *by the hypothesis of uniform impatience.*

2.4. **Constrained efficiency.** An *allocation* is a distribution of consumption plans across individuals. The space of allocations is

$$X = \left\{ x \in L\left(\mathbb{R}^J\right) : x^i \in X^i \text{ for every } i \in J \right\}.$$

An allocation x in X is *feasible* if it does not exceed aggregate resources and satisfies participation constraints, that is, for every t in $\mathbb{T}$,

$$\sum_{i \in J} x_t^i \leq \sum_{i \in J} e_t^i$$

and, for every individual i in J, for every t in $\mathbb{T}$,

$$U_t^i\left(x^i\right) \geq 0.$$

The space of all feasible allocations is denoted by $X(e)$. Notice that feasibility reflects both material constraints and participation constraints.

An allocation x in $X(e)$ is *Pareto constrained efficient* if it is not Pareto dominated by an alternative allocation z in $X(e)$. It is *Malinvaud constrained efficient* if it is not Pareto dominated by an alternative allocation z in $C(e, x)$, where $C(e, x)$ contains all allocations z in $X(e)$ such that

$$\left\{ t \in \mathbb{T} : \sum_{i \in J} \left| z_t^i - x_t^i \right| > 0 \right\} \text{ is finite.}$$

Clearly, any Pareto optimum is a Malinvaud optimum. However, Malinvaud optimality is a largely weaker requirement: For instance, any autarchic allocation is a Malinvaud optimum. Malinvaud constrained efficiency, inspired by Malinvaud [29, 30], is studied by Balasko and Shell [7], Aliprantis, Brown and Burkinshaw [3] and, recently, Bloise, Reichlin and Tirelli [8]: It captures the absence of any Pareto improvement, subject to material feasibility and participation, over any arbitrary finite horizon.

3. Equilibrium, Prices and Solvency Constraints

Trade occurs sequentially. In every period, a full spectrum of elementary Arrow securities is available for trade, each of which promising a unitary payoff, contingent on the occurrence of a distinct event in the following period. The asset market is, thus, sequentially complete. It simplifies to represent implicit prices of contingent commodities in terms of present values. They are denoted by p in P, the space of all strictly positive elements of $L(\mathbb{R})$, that is, at every t in $\mathbb{T}$, $\inf_{\omega \in \Omega} p_t(\omega) > 0$. In every period of trade t in $\mathbb{T}$, conditional on available information, a portfolio with deliveries v_{t+1} in $L_{t+1}(\mathbb{R})$ at the following date-events, has a market value, in terms of current consumption, given by

$$\text{current value of future deliveries } v_{t+1} = \frac{1}{p_t} \mathbb{E}_t p_{t+1} v_{t+1}.$$

It should be remarked that, at a price p in P, the present value of an arbitrary bounded consumption plan is not necessarily finite.

An individual i in J participates into financial markets. The holding of securities is represented by a *financial plan* v^i in V^i, the space of all unrestricted elements of $L(\mathbb{R})$. Positive values correspond to claims, whereas negative values are liabilities. This participation occurs subject to a *sequential budget constraint* imposing, in every period t in $\mathbb{T}$,

$$\mathbb{E}_t p_{t+1} v_{t+1}^i + p_t \left(x_t^i - e_t^i \right) \leq p_t v_t^i.$$

Accumulated wealth serves to finance current consumption, in excess to current endowment, and current net asset positions (claims or liabilities). Participation into financial markets is further restricted by quantitative limits to private liabilities. These *debt limits* are given by f^i in F^i, the set of all *positive* and *bounded* elements of $L(\mathbb{R})$. The financial plan v^i in V^i is subject, in every period of trade t in $\mathbb{T}$, to a *debt (or solvency) constraint* of the form

$$-f_t^i \leq v_t^i.$$

From the perspective of the individual, debt limits are given exogenously.

As in Eaton and Gersovitz [13], Kehoe and Levine [18], Kocherlakota [22] and Alvarez and Jermann [4], commitment is limited. Individuals might not honor their debt obligations, even though the material availability of future endowments would suffice for a complete repayment. When debt is repudiated, assets are seized and the individual is excluded from future participation into financial markets, though maintaining claims into future uncertain endowment. Thus, unhonored debt induces a permanent reverse to autarchy. At equilibrium, debt limits serve to guarantee that, on the one side, debt repudiation is not profitable for individuals and, on the other side, the maximum sustainable development of financial markets is enforced. This is the notion of equilibrium with *not-too-tight* debt constraints provided by Alvarez and Jermann [4].

Formally, an allocation x in X is an *equilibrium allocation* if there exist a price p in P, debt limits f in F and financial plans v in V satisfying the following properties:

(a) For every individual i in J, the plan $\left(x^i, v^i\right)$ in $X^i \times V^i$ is optimal subject to budget and debt constraints, given initial claims, where the optimal consumption plan is generated by

$$J_t^i \left(v_t^i; f^i \right) = \max V_t^i \left(x_t^i, J_{t+1}^i \left(v_{t+1}^i; f^i \right) \right)$$

subject to a budget constraint,

$$\mathbb{E}_t p_{t+1} v_{t+1}^i + p_t \left(x_t^i - e_t^i\right) \leq p_t v_t^i,$$

and a solvency constraint,

$$-f_{t+1}^i \leq v_{t+1}^i.$$

(b) Commodity and financial markets clear, that is, at every t in $\mathbb{T}$,

$$\sum_{i \in J} x_t^i = \sum_{i \in J} e_t^i \text{ and } \sum_{i \in J} v_t^i = 0.$$

(c) For every individual i in $\mathcal{J}$, debt limits are not-too-tight, that is, at every t in $\mathbb{T}$,

$$J_t^i \left(-f_t^i; f^i\right) = 0 = U_t^i \left(e^i\right).$$

Notice that, at equilibrium, for every individual i in J, in every period of trade t in $\mathbb{T}$,

$$U_t^i \left(x^i\right) = J_t^i \left(v_t^i; f^i\right) \geq J_t^i \left(-f_t^i; f^i\right) = 0.$$

Hence, an equilibrium allocation x in X is, as a matter of fact, an element of $X(e)$, the space of feasible allocations.

For the analysis of the equilibrium set, I move from the basic Welfare Theorems established by Bloise, Reichlin and Tirelli [8, Propositions 2-3]. A remarkable implication is that, in order to characterize the structure of competitive equilibrium, the set of Malinvaud constrained optima can be equivalently studied.

Welfare Theorems. *An allocation x in $X(e)$ is an equilibrium allocation if and only if it is Malinvaud constrained efficient.*

4. Recursive Method

4.1. **Overview.** In order to understand implications of Malinvaud constrained efficiency (and, hence, of competitive equilibrium under limited commitment), I construct an artificial conditional planner program. This hypothetical planner distributes current consumptions and continuation utilities so as to maximize the weighted sum of utilities subject to material feasibility and participation restrictions. At every contingency, the planning program is conditional on a *given* feasible set of continuation utilities and it generates a feasible set of utilities. At a fixed point, utility feasible sets are consistent over time and the conditional planner program delivers Malinvaud (but not necessarily Pareto) constrained efficiency.

4.2. **Feasible utility surpluses.** Adhering to Lucas and Stokey [27], I represent restrictions on utility surpluses (*i.e.*, excesses upon autarchic utilities) in terms of support functions. Let Θ be the canonical positive simplex in $\mathbb{R}^J$, that is,

$$\Theta = \left\{ \theta \in \mathbb{R}_+^J : \sum_{i \in J} \theta^i = 1 \right\}.$$

For a positive continuous convex map $f : \Theta \to \mathbb{R}$,

$$V_f = \left\{ v \in \mathbb{R}_+^J : f(\theta) \geq \theta \cdot v \text{ for every } \theta \in \Theta \right\}$$

is a convex compact set. On the other side, given a (non-empty) convex compact set V of $\mathbb{R}_+^J$,

$$f_V(\theta) = \max_{v \in V} \theta \cdot v$$

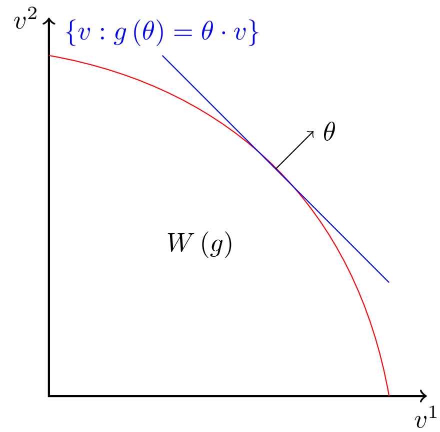

FIGURE 1. Feasible utilities

is a positive continuous convex map. Basically, V_f is interpreted as a set of feasible utility surpluses, whereas $f(\theta)$ represents the maximum social welfare, subject to feasibility, at welfare weights θ in Θ. The geometric construction is illustrated in figure 1.

4.3. **Conditional planner program.** Feasible welfare gains evolve over time subject to uncertainty. To represent this stochastic evolution, I introduce the space G, consisting of all elements g of $L^\infty(C(\Theta))$ such that, for every t in $\mathbb{T}$, at every ω in Ω, $g_t(\omega) : \Theta \to \mathbb{R}$ is a positive continuous convex map. (Here $C(\Theta)$ denotes the linear space of all continuous maps $f : \Theta \to \mathbb{R}$, endowed with the supremum norm.) The induced feasible set for utility surpluses is given by

$$W_t(g) = \left\{w_t \in L_t\left(\mathbb{R}^J_+\right) : g_t(\theta) \geq \theta \cdot w_t \text{ for every } \theta \in \Theta\right\}.$$

Notice that the construction describes a convex compact set contingent on the state of nature ω in Ω.

Take any arbitrarily given map g in G. At every t in $\mathbb{T}$, contingent on available information, a planner distributes current consumptions x_t in $X_t = L_t\left(\mathbb{R}^J_+\right)$ and future welfare surpluses w_{t+1} in $W_{t+1}(g)$. The feasible set $F_t(g)$ contains all (x_t, w_{t+1}) in $X_t \times W_{t+1}(g)$ satisfying material feasibility,

$$\sum_{i\in J} x^i_t - \sum_{i\in J} e^i_t \leq 0, \tag{F-1}$$

and participation, for every individual i in J,

$$V^i_t\left(x^i_t, w^i_{t+1}\right) \geq 0. \tag{F-2}$$

This conditional planner program generates, as a value function, an alternative map $\hat{g}$ in G.

Formally, the (Bellman) operator $T : G \to G$ is defined, at every t in $\mathbb{T}$, by

$$(Tg)_t(\theta) = \max_{(x_t, w_{t+1}) \in F_t(g)} \theta \cdot V_t(x_t, w_{t+1}).$$

Notice that this social planner problem is imposed, subject to measurability, at every state ω in Ω. Relevantly, the operator $T : G \to G$ decomposes as a sequence of operators $T_t : G_{t+1} \to G_t$, using the obvious identification $G = \oplus_{t\in\mathbb{T}} G_t$. As

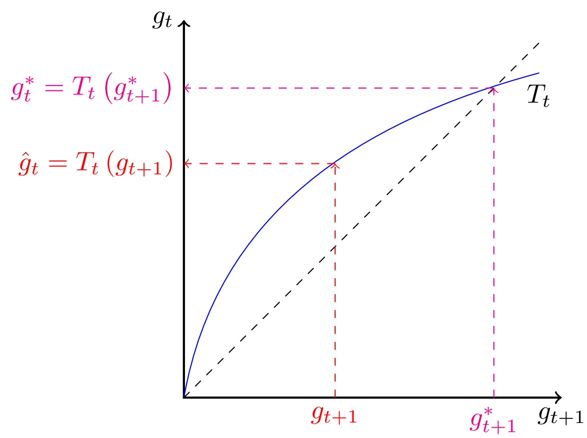

FIGURE 2. Monotone concavity of the operator

a matter of fact, at every period t in $\mathbb{T}$, the value of the social planning problem $(Tg)_t$ in G_t only depends on the restrictions on utility surpluses induced by g_{t+1} in G_{t+1}. Adhering to the same convention, at every t in $\mathbb{T}$, notation can be simplified: The feasible set for welfare gains only depends on g_t in G_t, $W_t\left(g_t\right)$, and the overall feasible set only depends on g_{t+1} in G_{t+1}, $F_t\left(g_{t+1}\right)$. A heuristic representation of the Bellman operator is given in figure 2.

4.4. **Properties.** The operator is *not* a contraction. The failure of the contraction property is immediately understood: even though individuals are impatient, from an additional social value as continuation utilities, the planner might extract a more than proportional current social value, as this renders more permissive participation constraints. Nevertheless, the operator fulfills two fundamental properties. First, it is monotone, as more lenient restrictions on continuation utilities increase current social value. Second, it is concave, as the feasible set is convex and social welfare is strictly concave in current consumptions and linear in continuation utility surpluses. These two fundamental properties will be heavily exploited in the analysis.

Lemma 1 (Basic properties)**.** *The operator $T : G \to G$ is well-defined. Furthermore, $T(G) \subset G^*$, where G^* contains all maps g in G satisfying, at every t in $\mathbb{T}$, given any ω in Ω,*

$$\max_{\theta \in \Theta} g_t(\omega)(\theta) > 0 \text{ only if } \min_{\theta \in \Theta} g_t(\omega)(\theta) > 0 \tag{G-1}$$

and

$$g_t(\omega)(\theta) = \max_{w_t \in W_t(g)} \theta \cdot w_t(\omega). \tag{G-2}$$

Finally, the operator is (weakly) increasing and (weakly) concave.

Property (G-1) guarantees that welfare gains are uniformly strictly positive whenever they do not vanish. Property (G-2) establishes that the social value is always attained by some feasible utility gains. To verify continuity of the operator, and for other results in the following analysis, I prove some preliminary technical lemmas.

Lemma 2 (Equicontinuity)**.** *Given any g^* in G, there exists $\kappa > 0$ such that, at every t in $\mathbb{T}$, given any g in $\{g \in G : g \leq g^*\}$,*

$$\left\| (Tg)_t (\theta) - (Tg)_t (\theta^*) \right\|_t \leq \kappa \left\| \theta - \theta^* \right\|.$$

Lemma 3 (Closure)**.** *Given any map g^* in G,*

$$closure\left(T\left(\{g \in G : g \leq g^*\}\right)\right) \subset G^*,$$

where the closure is taken in the (relative) product topology.

The above lemmas enable me to show continuity of the Bellman operator in the (relative) product topology. Notice that this is established on a truncated domain. An alternative approach could exploit monotonicity and verify continuity along increasing, or decreasing, sequences.

Proposition 1 (Continuity)**.** *Given any map g^* in G, the restricted operator $T : \{g \in G^* : g \leq g^*\} \to G^*$ is continuous in the (relative) product topology.*

4.5. **Recursive generation.** A fixed point g in G of the Bellman operator is said to be a *support map*. I now explain how to relate support maps and feasible allocations. This exploits the basic principles of dynamic programming.

An allocation x in $X(e)$ is *consistent* with support map g in G if, at every t in $\mathbb{T}$,

$$\min_{\theta \in \Theta} \left[g_t(\theta) - \theta \cdot U_t(x)\right] = 0,$$

where it is understood that the minimum is taken point-wise at every ω in Ω. Thus, consistency requires that the utility gains, generated by the given allocation, fulfill the restrictions prescribed by the support map and, for some welfare weights, maximize social welfare. In geometrical terms, consistent allocations generate utility surpluses, at every contingency, on the frontier of the feasible set for utility surpluses defined by the support map.

I begin with a preliminary lemma, showing that any current utility surpluses in the feasible set are attained by a feasible distribution of consumptions and of future utility surpluses in the feasible set at the following contingencies.

Lemma 4 (Feasible utilities)**.** *Given any map g in G, at every t in $\mathbb{T}$,*

$$v_t \in W_t\left(T_t\left(g_{t+1}\right)\right) \text{ if and only if } v_t \in V_t\left(F_t\left(g_{t+1}\right)\right).$$

I can now illustrate the basic relation between support maps and consistent allocations. Basically, any support map generates a multiplicity of consistent allocations by varying initial welfare weights. The same procedure might be applied at any arbitrary contingency, reinterpreted as the initial node of the economy.

Proposition 2 (Recursive generation)**.** *Given a support map g in G, for every welfare weights θ in Θ, there exists an allocation x in $X(e)$, which is consistent with support function g in G, satisfying*

$$g_0(\theta) = \theta \cdot U_0(x).$$

It is worth remarking that any consistent allocation can be represented as an adapted process on the domain Θ, the space of welfare weights. The evolution of welfare weights, indeed, corresponds to movements, across contingencies and time, along the Pareto frontiers of feasible utility gains generated by the support map. Welfare weights also account for the distribution of current consumptions across individuals.

5. Self-generation

I preliminarily observe that the operator admits two obvious fixed points. First, the autarchic fixed point, defined by $g_t = 0$ at every t in $\mathbb{T}$. Indeed, when no utility gains can be distributed, because of participation restrictions, the autarchic allocation is the only solution to the conditional planner problem. Second, the constrained optimal fixed point, which is defined, at every date-event t in $\mathbb{T}$, by

$$g_t^* (\theta) = \max_{x \in X(e)} \theta \cdot U_t (x) ,$$

where, as usual, it is understood that the maximum is taken point-wise at every state ω in Ω. In other terms, this support map is the value of the unrestricted planner problem, under material feasibility and participation. It is a fixed point by a canonical application of the Principle of Optimality. These two fixed points are *extreme*, in the sense that any other support map g in G is restricted by

$$0 \leq g \leq g^*.$$

In geometric terms, any support map generates feasible sets of utility gains, across time and contingencies, that are dominated by those corresponding to constrained efficiency.

I now show that support maps allow for a complete characterization of Malinvaud constrained optima. In particular, an allocation is consistent with some support map only if it is Malinvaud constrained efficient and any Malinvaud constrained efficient allocation is consistent with some support map. The former claim is almost immediate, whereas the latter deserves some elaboration.

In order to establish that an allocation is consistent with a support map only if it is Malinvaud constrained efficient, it suffices to exploit the Principle of Optimality restricted to finite, though indefinite, horizons. The proof unfolds by contradiction. Indeed, supposing not, a consistent allocation is strictly Pareto dominated by an alternative feasible allocation involving a redistribution only over some finite horizon. As this reallocation ceases at some period, utility surpluses are in the feasible set of the planner program for any sufficiently large period. Hence, in the previous period, the given allocation cannot be improved by any feasible redistribution. This reveals a contradiction by backward induction.

Proposition 3 (Generation)**.** *An allocation x in $X(e)$ is consistent with some support map g in G only if it is Malinvaud constrained efficient.*

The proof that any Malinvaud constrained efficient allocation is consistent with some support map relies on a constructive reasoning. For a given Malinvaud constrained efficient allocation, define a restricted planner program by allowing feasible readjustments in consumption only over finite horizons (a similar argument is adopted, for different purposes, by Bloise, Reichlin and Tirelli [8]). Exploiting the Principle of Optimality, it is showed that the value function of this restricted planner program, at every contingency, stabilizes the Bellman operator.

Proposition 4 (Exhaustion)**.** *An allocation x in $X(e)$ is consistent with some support map g in G if it is Malinvaud constrained efficient.*

The established coincidence permits the study of Malinvaud constrained efficiency by means of properties of the set of support maps (*i.e.*, fixed points of the Bellman operator). The advantage of this indirect method relies on the fact that monotone concavity by itself imposes a peculiar form to the set of fixed points. The remaining part of this manuscript is devoted to uncover these regularities.

6. Characterization

6.1. **Markov intrinsic uncertainty.** For the purpose of characterization, I assume that the only uncertainty affecting fundamentals (preferences and endowments) is generated by a Markov process. In particular, restrictions are of a twofold nature: the Markov state space is finite and the economy is indecomposable (all Markov states are reachable from any given state). Markov intrinsic uncertainty imposes further discipline on the Bellman operator and ensures a strong form of concavity. It is redundant to notice that other sources of extrinsic uncertainty might still be of real allocative relevance, even though unrelated to fundamentals.

The Markov transition map is $\bar{\Pi} : \bar{S} \to \Delta\left(\bar{S}\right)$, with $\bar{S}$ being a finite set and $\Delta\left(\bar{S}\right)$ representing the space of probability measures on $\bar{S}$. This Markov chain is indecomposable, that is, given any s in $\bar{S}$, for every s^* in $\bar{S}$, $\bar{\Pi}_s^n\left(s^*\right) > 0$ for some n in $\mathbb{N}$. Uncertainty is consistent with this Markov process. In other terms, there exists an adapted process $\bar{\sigma}$ in $L\left(\bar{S}\right)$ satisfying

$$\mu\left(\bar{\sigma}_{t+1} = s^* | \bar{\sigma}_t = s\right) = \bar{\Pi}_s\left(s^*\right).$$

The natural interpretation is that, at every state of nature ω in Ω, the current Markov state $\bar{\sigma}_t\left(\omega\right)$ in $\bar{S}$ conveys all relevant information about fundamentals (preferences and endowments) in the continuation economy.

The space $\mathcal{M}\left(G\right)$ consists of all *Markov maps*, that is, it contains all maps g in G such that, for some $\vec{g} : \bar{S} \to C\left(\Theta\right)$,

$$g_t\left(\omega\right) = \vec{g} \circ \bar{\sigma}_t\left(\omega\right).$$

It is relevant to observe that $\mathcal{M}\left(G\right)$ is *invariant* for the Bellman operator, that is, $T\left(\mathcal{M}\left(G\right)\right) \subset \mathcal{M}\left(G\right)$. This is an immediate implication of the fact that fundamentals are only affected by Markov states and, thus, the planning program is substantially the same at contingencies corresponding to the identical Markov state, provided that continuation utilities themselves only reflect Markov intrinsic uncertainty.

6.2. **Markov uniqueness.** As the economy is Markovian, constrained efficient allocations only reflect Markov uncertainty. In particular, the constrained-efficient support map g^* in G is necessarily Markov (*i.e.*, an element of $\mathcal{M}\left(G\right)$). This permits to establish a strong form of concavity of the Bellman operator.

Lemma 5 (Strict concavity)**.** *For some sufficiently large n in $\mathbb{N}$, given any λ in $(0,1)$, there exists $f\left(\lambda\right)$ in $\left(\lambda, 1\right)$ such that*

$$T^n\left(\lambda g^*\right) \geq f\left(\lambda\right) g^*.$$

The intuition is straightforward. For every contraction of the feasible sets of future utility gains, a proportional contraction of consumptions towards autarchy is feasible in the conditional planning program. Because of strict concavity of per-period utilities, this guarantees strict concavity of the Bellman operator whenever constrained efficient consumptions do not coincide with autarchy. This is necessarily the case in an indecomposable economy over some sufficiently large horizon, which explains the need for iterations of the Bellman operator. Finally, Markov intrinsic uncertainty ensures uniformity in strict concavity, as there are only finitely many Markov states.

The important implication of strict concavity is that the only Markov support maps correspond to constrained efficiency and to autarchy. It is worth remarking the relevant role of ergodicity, more than of a finite state space, in delivering uniqueness. Removing ergodicity, there are certainly other non-autarchic Markov maps which are constrained inefficient. For instance, it is simple to verify existence of non-autarchic support maps with the property that, from some Markov states, there is a transition to an *absorbing* autarchic Markov state. As far as terminology is concerned, a map g in G is uniformly strictly positive if there exists $\epsilon > 0$ such that, at every t in $\mathbb{T}$, $\inf_{\theta \in \Theta} g_t(\omega)(\theta) \geq \epsilon$ for every state of nature ω in Ω. Notice that, by the hypothesis of irreducibility, g^* in G is uniformly strictly positive whenever autarchy is constrained inefficient.

Proposition 5 (Uniqueness)**.** *Any uniformly strictly positive support map g in G is constrained efficient. In particular, any non-autarchic Markov support map g in $\mathcal{M}(G)$ is constrained efficient.*

Uniqueness bears remarkable implications for competitive equilibrium. Markov support maps correspond to competitive equilibria in which variables obey a Markov transition on the state space $\bar{S} \times \Theta$. Roughly speaking, $\bar{S}$ accounts for uncertainty related to fundamentals, whereas Θ reflects the evolution of the distribution of wealth across individuals. According to proposition 5, those competitive equilibria are constrained efficient. Nothing prevents the existence of constrained inefficient equilibria on a larger Markov space (including, for instance, some sunspot signal).

6.3. **Computation.** For computational purposes, it is worth examining conditions guaranteeing convergence of canonical algorithms. I here show that the iterations of the Bellman operator converge to constrained efficiency for any arbitrarily specified initial map, provided that this is uniformly strictly positive. This represents a substantial improvement with respect to previous results: Alvarez and Jermann [5] prove convergence only when the algorithm is initiated at unconstrained efficiency, that is, at the value function of a planner program without participation constraints. It guarantees reliability of traditional computational methods.

Proposition 6 (Computation)**.** *When autarchy is constrained inefficient, given any uniformly strictly positive Markov map g in $\mathcal{M}(G)$, the sequence $(T^n(g))_{n\in\mathbb{N}}$ in $\mathcal{M}(G)$ converges uniformly to the constrained efficient support map g^* in $\mathcal{M}(G)$.*

6.4. **Basic multiplicity.** A fundamental multiplicity of support maps is established by exploiting the property of monotone concavity of the Bellman operator. Basically, this implies the existence of a monotonically increasing orbit connecting the two extreme fixed points, autarchy and constrained efficiency, as graphically illustrated by figure 3. Notice that only the limits of this orbit are support maps, that is, fixed points of the Bellman operator. The proof is an adaptation of a theorem by Dancer and Hess [11] (see also Hirsch and Smith [16] on monotone dynamics and sublinear operators).

Lemma 6 (Connecting orbit)**.** *There exists a monotonically increasing orbit $(g^n)_{n\in\mathbb{Z}}$ in G, with $g^{n+1} = T(g^n)$ for every n in $\mathbb{Z}$, such that $(g^n)_{n\in\mathbb{N}}$ and $(g^{-n})_{n\in\mathbb{N}}$ converge uniformly to, respectively, the constrained efficient and the autarchic support*

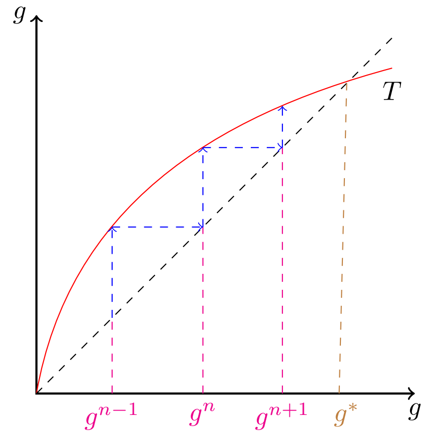

FIGURE 3. Connecting orbit

maps in G. Furthermore, for every arbitrary λ_0 in $(0,1)$, there exists such an orbit satisfying

$$\lambda_0 = \sup\left\{\lambda \in [0,1] : g^0 \geq \lambda g^*\right\}.$$

The connecting orbit reveals a basic multiplicity: There exists a *continuum* of support maps g in G with the property that, for every ω in Ω, $g_t(\omega)$ monotonically vanish over time t in $\mathbb{T}$. This is immediate once one realizes that support maps in G can be constructed, by means of reverse diagonalization, along the orbit. Indeed, at every t in $\mathbb{T}$,

$$g_t^{-t} = T\left(g^{-(t+1)}\right)_t = T_t\left(g_{t+1}^{-(t+1)}\right).$$

Thus, in a sense, the backward branch of the orbit might be reinterpreted as the dynamic evolution of a support map over time and uncertainty. Intuitively, the Bellman operator accounts for backward dynamics at equilibrium, as it maps future economic variables into current economic variables.

Proposition 7 (Basic multiplicity)**.** *There exists a* continuum *of support maps g in G which monotonically vanish over time,* i.e., *at every ω in Ω,*

$$\lim_{t\in\mathbb{T}} \left\|g_t(\omega)\right\|_\infty = 0.$$

In the case of no uncertainty, this characterization is exhaustive: there exists a stationary constrained efficient support map and a *continuum* of non-stationary support maps monotonically converging, over time, to autarchy. This structure of completive equilibria resembles that of economies with overlapping generations under gross substitutability, as studied by Kehoe, Levine, Mas-Colell and Woodford [20]. Under uncertainty, the dynamics might be more complex. The remaining part of the analysis is devoted to clarify the necessity of a long-run market collapse when constrained efficiency fails.

6.5. **Ergodicity.** In this last part of the analysis, I study support maps which are Markovian on some extended state space admitting an ergodic invariant measure. Though the formulation is more general, it is of help to refer to the following situation: current over-all Markov state consists of a Markov state related to fundamentals and an additional sunspot shock; sunspot shocks are identically and independently distributed over time; the evolution of fundamental Markov states is governed by a Markov chain; consumptions incorporate both intrinsic and extrinsic uncertainty. I show that the only non-autarchic extended Markov support map is constrained efficiency. Again, removing ergodicity, uniqueness would not be preserved.

Given a topological space S, endowed with its Borel σ-algebra, a Markov transition is a (measurable) map $\Pi : S \to \Delta(S)$, where $\Delta(S)$ denotes the space of all probability measures on S. I impose some regularity conditions on the Markov transition.

Markov transition (Compact state space)**.** *The state space S is a compact metric space.*

Markov transition (Strong Feller Property)**.** *For every measurable set E of S, $s \mapsto \Pi_s(E)$ is a continuous map.*

For convenience, I here recollect some common concepts for Markov transitions. A probability measure π in $\Delta(S)$ is invariant if, for every measurable set E of S,

$$\pi(E) = \int_S \Pi_s(E)\, d\pi(s).$$

A measurable set E of S is *invariant* if $\Pi_s(E) = 1$ for every s in E. An invariant probability measure π in $\Delta(S)$ is *ergodic* if, for every invariant set E of S, either $\pi(E) = 0$ or $\pi(E) = 1$. Finally, given an ergodic invariant probability measure π in $\Delta(S)$, an invariant set E of S is ergodic if $\pi(E) = 1$. It is well-known that distinct ergodic invariant probability measures must be mutually singular (see, for instance, Hairer [14, Theorem 1.7]). An important implication of the Strong Feller Property is that the topological supports of any two mutually singular invariant probability measures are disjoint (for instance, Hairer [14, Proposition 2.6]).

I assume that uncertainty is consistent with the Markov transition $\Pi : S \to \Delta(S)$, that is, there exists an adapted process σ in $L(S)$ such that, for every measurable set E of S,

$$\mu(\sigma_{t+1} \in E | \sigma_t = s) = \Pi_s(E).$$

Clearly, at every state ω in Ω, $\sigma_t(\omega)$ is interpreted as the Markov state in S prevailing at time t in $\mathbb{T}$. Extrinsic uncertainty consistently expands intrinsic uncertainty, in the sense that there exists a continuous map $\varphi : S \to \bar{S}$ such that $\bar{\sigma} = \varphi \circ \sigma$, where $\bar{S}$ in $L(\bar{S})$ is the adapted process representing the evolution of Markov states related to fundamentals. Finally, I assume that, for some ergodic invariant measure π in $\Delta(S)$, $\mathring{S} \subset S$ is an ergodic set (*i.e.*, it is invariant with $\pi\left(\mathring{S}\right) = 1$) and contains no smaller invariant set (*i.e.*, if $E \subset \mathring{S}$ is an invariant set, then $\mathring{S} \subset E$), where

$$\mathring{S} = \bigcup_{t \in \mathbb{T}} \{s \in S : \sigma_t(\omega) = s \text{ for some } \omega \in \Omega\}.$$

In other terms, the Markov state space is no larger than necessary for an ergodic stochastic process.

A map g in G is *ergodic* if there exists a continuous map $\vec{g} : S \to C(\Theta)$ such that, at every t in $\mathbb{T}$,

$$g_t(\omega) = \vec{g} \circ \sigma_t(\omega).$$

The interpretation is that variables of economic interest are determined by some Markov process, which might be related or unrelated to fundamentals. In other terms, current Markov state completely determines all economic variables, whereas Markov transitions govern dynamics. The space of all such ergodic maps in denoted by $\mathcal{E}(G)$. Clearly, $\mathcal{E}(G)$ is an invariant set for the Bellman operator, $T(\mathcal{E}(G)) \subset \mathcal{E}(G)$.

Proposition 8 (Ergodicity)**.** *Under the stated conditions, any non-autarchic ergodic support map g in $\mathcal{E}(G)$ is constrained efficient.*

The intuition for this property is simple. When constrained efficiency fails at some contingency, the economy will be arbitrarily close to autarchy with positive probability in the long-run. Hence, autarchy is an accumulation point of the ergodic invariant set. However, under the strong Feller property, supports of distinct ergodic invariant measures are disjoint. At the same time, the autarchy itself is an ergodic invariant set. Hence, when ergodicity is imposed, constrained efficiency fails only if the economy collapses to autarchy immediately.

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Proofs

Proof of lemma 1. At every t in $\mathbb{T}$, it is simple to prove that the feasible set $F_t(g)$ is non-empty, convex and compact. Hence, by Berge's Maximum Theorem [2, Theorem 17.31], for every ω in Ω, $(Tg)_t(\omega) : \Theta \to \mathbb{R}$ is a continuous map. It is also positive and, by canonical arguments, convex. Measurability of $(Tg)_t : \Omega \to L_t(C(\Theta))$ is also an obvious property. For boundedness, observe that, at every t in $\mathbb{T}$, given any feasible plan (x_t, w_{t+1}) in $F_t(g)$,

$$\theta \cdot V_t(x_t, w_{t+1}) \leq \sup_{i \in J} \left[\sup_{c \in \mathbb{R}_+} \left| u^i(c) \right| - u^i(0) \right] + \frac{1}{\eta} \|g\|_\infty ,$$

where $1 > \eta > 0$ is given by the hypothesis of uniform impatience. This establishes that the operator maps G into itself. Finally, in order to verify that property (G-1) is fulfilled, simply observe that current consumptions might be redistributed without varying continuation utilities and that preferences are strictly monotone in consumption. For property (G-2), suppose that $\left(x_t^*, w_{t+1}^*\right)$ in $F_t(g)$ achieves the maximum at welfare weights θ^* in Θ and let $w_t^* = V_t\left(x_t^*, w_{t+1}^*\right) \geq 0$ in $L_t\left(\mathbb{R}^J\right)$. By construction, $(Tg)_t(\theta^*) = \theta^* \cdot w_t^*$ and, by optimality, as the plan $\left(x_t^*, w_{t+1}^*\right)$ lies in the feasible set $F_t(g)$, $(Tg)_t(\theta) \geq \theta \cdot w_t^*$, for every θ in Θ. Monotonicity and concavity are implied by canonical arguments. In particular, notice that, given any (g, g') in $G \times G$, for every $1 > \lambda > 0$, $\lambda W_t(g) + (1-\lambda) W_t(g') \subset W_t(\lambda g + (1-\lambda) g')$ at every t in $\mathbb{T}$. This suffices to deliver $T(\lambda g + (1-\lambda) g') \geq \lambda T(g) + (1-\lambda) T(g')$. □

Proof of lemma 2. Define

$$\kappa = \sup_{i \in J} \left[\sup_{c \in \mathbb{R}_+} \left| u^i(c) \right| - u^i(0) \right] + \frac{1}{\eta} \|g^*\|_\infty ,$$

where $1 > \eta > 0$ is given by the hypothesis of uniform impatience. Notice that, given any feasible plan (x_t, w_{t+1}) in $F_t(g_{t+1})$, for every individual i in J,

$$\begin{aligned} V_t^i\left(x_t^i, w_{t+1}^i\right) &\leq \left| u^i\left(x_t^i\right) - u^i\left(e_t^i\right) \right| + \frac{1}{\pi_t^i} \mathbb{E}_t \pi_{t+1}^i w_{t+1}^i \\ &\leq \sup_{c \in \mathbb{R}_+} \left| u^i(c) \right| - u^i(0) + \frac{1}{\pi_t^i} \mathbb{E}_t \pi_{t+1}^i \|g\|_\infty \\ &\leq \sup_{c \in \mathbb{R}_+} \left| u^i(c) \right| - u^i(0) + \frac{1}{\eta} \|g^*\|_\infty . \end{aligned}$$

It follows that, given any g in $\{g \in G : g \leq g^*\}$, at every t in $\mathbb{T}$,

$$\begin{aligned} (Tg)_t(\theta) &= \theta \cdot V_t(x_t, w_{t+1}) \\ &\leq (Tg)_t(\theta^*) + (\theta - \theta^*) \cdot V_t(x_t, w_{t+1}) \\ &\leq (Tg)_t(\theta^*) + \kappa \|\theta - \theta^*\| , \end{aligned}$$

where (x_t, w_{t+1}) is the feasible plan in $F_t(g_{t+1})$ achieving the maximum at welfare weights θ in Θ. Reversing the role of alternative welfare weights, one obtains

$$\left\| (Tg)_t(\theta) - (Tg)_t(\theta^*) \right\|_t \leq \kappa \|\theta - \theta^*\| ,$$

thus proving the claim. □

Proof of lemma 3. The only property to verify is (G-1), as the other properties are certainly satisfied. Consider any sequence $(g^n)_{n \in \mathbb{N}}$ in $T(\{g \in G : g \leq g^*\})$ converging to g in the closure of $T(\{g \in G : g \leq g^*\})$. For every n in $\mathbb{N}$, there exists h^n in $\{g \in G : g \leq g^*\}$ such that $g^n = T(h^n)$. Fix any period t in $\mathbb{T}$. Also, peg any state ω in Ω and let the following argument be conditional on the event $\mathcal{F}_t(\omega)$ in $\mathcal{F}_t$. Suppose that, for some θ^* in Θ, $g_t(\theta^*) > \epsilon > 0$. By convergence, $g_t^n(\theta^*) > \epsilon$ for every sufficiently large n in $\mathbb{N}$ and, hence, at no loss of generality, for every n in $\mathbb{N}$. Thus, for every n in $\mathbb{N}$, there exists a feasible plan $\left(x_t^n, w_{t+1}^n\right)$ in $F_t(h^n)$ satisfying $\theta^* \cdot V_t\left(x_t^n, w_{t+1}^n\right) > \epsilon$. Let η_t^n in X_t be defined by

$$\eta_t^{in} = \sup\left\{ \eta \geq 0 : V_t^i\left(x_t^{in} - \eta, w_{t+1}^i\right) \geq 0 \right\} .$$

Finally, set z_t^n in X_t by

$$z_t^{in} = x_t^{in} - \eta_t^{in} + \left(\frac{\sum_{j \in J} \eta_t^{jn}}{\operatorname{card}(J)} \right) .$$

By feasibility of this alternative plan, it follows that, at every n in $\mathbb{N}$, $g_t^n(\theta) \geq \theta \cdot V_t\left(z_t^n, w_{t+1}^n\right)$ for every θ in Θ. At no loss of generality, the sequence $\left(z_t^n, w_{t+1}^n\right)_{n \in \mathbb{N}}$ in $L_t\left(\mathbb{R}^J\right) \times L_{t+1}\left(\mathbb{R}^J\right)$ converges, as it lies in a compact set. Hence,

$$\lim_{n \to \infty} g_t^n(\theta) = 0 \text{ only if } \lim_{n \to \infty} \sum_{j \in J} \eta_t^{jn} = 0 \text{ only if } \lim_{n \to \infty} g_t^n(\theta^*) = 0,$$

which reveals a contradiction. This suffices to prove the claim. □

Proof of proposition 1. To verify the claim about continuity, peg any t in $\mathbb{T}$ and let the argument be conditional on some state of nature ω in Ω (or on the minimal event $\mathcal{F}_t(\omega)$ in $\mathcal{F}_t$). The feasible correspondence might be restricted (in both domain and range) as

$$F_t : \left\{g_{t+1} \in G^*_{t+1} : g_{t+1} \leq g^*_{t+1}\right\} \hookrightarrow X^*_t \times W_{t+1}\left(g^*_{t+1}\right),$$

where

$$X^*_t = \left\{x_t \in X_t : \sum_{i \in J} x^i_t \leq \sum_{i \in J} e^i_t\right\}.$$

The range of this restricted correspondence is compact. To apply Berge's Maximum Theorem [2, Theorem 17.31], one needs to show that this correspondence is continuous with non-empty compact values. Non-empty compact values are obvious. Also, upper hemicontinuity follows from the fact that the graph is closed. Hence, it remains to verify lower hemicontinuity. To this purpose, peg any (x_t, w_{t+1}) in $F_t(g_{t+1})$. Also, choose any sequence $(g^n)_{n\in\mathbb{N}}$ in $\{g \in G^* : g \leq g^*\}$ converging to g in $\{g \in G^* : g \leq g^*\}$. Given any sufficiently small $\epsilon > 0$, define

$$\lambda^n = \left(1 - \frac{1}{\epsilon}\left\|g_{t+1} - g^n_{t+1}\right\|_{t+1}\right)^+.$$

Clearly, the sequence $(\lambda^n)_{n\in\mathbb{N}}$ of $[0,1]$ converges to the unity. Assume that $\epsilon > 0$ satisfies, at every state ω in Ω,

$$\max_{\theta\in\Theta} g_{t+1}(\omega)(\theta) > 0 \text{ only if } \min_{\theta\in\Theta} g_{t+1}(\omega)(\theta) \geq \epsilon.$$

Such a value exists because of property (G-1), along with measurability with respect to a finitely-generated σ-algebra. Take any ω in Ω with $\max_{\theta\in\Theta} g_{t+1}(\omega)(\theta) > 0$; for every n in $\mathbb{N}$,

$$g_{t+1}(\omega)(\theta) - g^n_{t+1}(\omega)(\theta) \leq \frac{1}{\epsilon}\left\|g_{t+1} - g^n_{t+1}\right\|_{t+1} g_{t+1}(\omega)(\theta);$$

hence, for all welfare weights θ in Θ,

$$g^n_{t+1}(\omega)(\theta) \geq \lambda^n g_{t+1}(\omega)(\theta) \geq \lambda^n \theta \cdot w_{t+1}(\omega).$$

Take any date-event ω in Ω with $\max_{\theta\in\Theta} g_{t+1}(\omega)(\theta) = 0$; for every n in $\mathbb{N}$,

$$g^n_{t+1}(\omega)(\theta) \geq 0 \geq \lambda^n g_{t+1}(\omega)(\theta) \geq \lambda^n \theta \cdot w_{t+1}(\omega).$$

To complete the proof, simply observe that, for every n in $\mathbb{N}$,

$$\left(e_t + \lambda^n(x_t - e_t), \lambda^n w_{t+1}\right) \in F_t\left(g^n_{t+1}\right),$$

which suffices to show lower hemicontinuity. Applying Berge's Maximum Theorem, it follows that, for every θ in Θ, the sequence $\left((Tg^n)_t(\theta)\right)_{n\in\mathbb{N}}$ in $L_t(\mathbb{R})$ converges to $(Tg)_t(\theta)$ in $L_t(\mathbb{R})$.

To show uniform convergence, I exploit lemma 2. For any arbitrarily chosen $\epsilon > 0$, there exists a finite subset Θ^* of Θ such that, for every θ in Θ,

$$\min_{\theta^*\in\Theta^*} \|\theta - \theta^*\| < \frac{\epsilon}{3\kappa}.$$

Hence, for any arbitrary θ in Θ,

$$\begin{aligned} \left\|(Tg)_t(\theta) - (Tg^n)_t(\theta)\right\|_t \leq\ & \left\|(Tg)_t(\theta) - (Tg)_t(\theta^*)\right\|_t \\ & + \left\|(Tg)_t(\theta^*) - (Tg^n)_t(\theta^*)\right\|_t \\ & + \left\|(Tg^n)_t(\theta^*) - (Tg^n)_t(\theta)\right\|_t, \end{aligned}$$

where the norm refer to the linear space $L_t(\mathbb{R})$. By the above argument, for every sufficiently large n in $\mathbb{N}$, the middle term does not exceed $(\epsilon/3)$. The same is true of the first and third term, because of lemma 2, for an appropriate choice of θ^* in Θ. This suffices to show that, given any $\epsilon > 0$, for every sufficiently large n in $\mathbb{N}$,

$$\left\|(Tg^n)_t - (Tg)_t\right\|_t < \epsilon,$$

where now the norm refers to the linear space $L_t(C(\Theta))$, thus proving the claim. □

Proof of lemma 4. One direction is obvious. So, consider any v_t in $W_t(T_t(g_{t+1}))$ and suppose that it does not belong to the non-empty compact set $V_t(F_t(g_{t+1}))$. Observe that, if (x_t, w_{t+1}) lies in $F_t(g_{t+1})$, then any $\left(x_t^*, w_{t+1}^*\right)$ in $X_t \times W_{t+1}$, with $x_t^* \leq x_t$ and $w_{t+1}^* \leq w_{t+1}$, is also in $F_t(g_{t+1})$, provided that $V_t\left(x_t^*, w_{t+1}^*\right) \geq 0$. By continuity and monotonicity of per-period utilities, this shows that

$$w_t \in V_t(F_t(g_{t+1})) \text{ if and only if } \left\{v_t \in L_t\left(\mathbb{R}^J\right) : 0 \leq v_t \leq w_t\right\} \subset V_t(F_t(g_{t+1})).$$

Exploiting this latter property, along with the concavity of per-period utilities, it is simple to show that $V_t(F_t(g_{t+1}))$ is a convex set. Also,

$$\left(v_t + L_t\left(\mathbb{R}_+^J\right)\right) \cap V_t(F_t(g_{t+1})) = \emptyset.$$

By the Strong Separation Theorem [2, Theorem 5.79], there exists a non-null θ in $L_t\left(\mathbb{R}^J\right)$ satisfying, for every (x_t, w_{t+1}) in $F_t(g_{t+1})$,

$$\theta \cdot v_t > \theta \cdot V_t\left(x_t, w_{t+1)}\right).$$

Also, θ is a positive element of $L_t\left(\mathbb{R}^J\right)$ and, hence, at no loss of generality, it is an element of $L_t(\Theta)$. This yields a contradiction, as, for every feasible (x_t, w_{t+1}) in $F_t(g_{t+1})$,

$$T_t(g_{t+1})(\theta) \geq \theta \cdot v_t > \theta \cdot V_t(x_t, w_{t+1}),$$

thus proving the claim. □

Proof of proposition 2. At every t in $\mathbb{T}$, peg any positive w_t in $L_t\left(\mathbb{R}_+^J\right)$ satisfying

$$\min_{\theta \in \Theta}\left[g_t(\theta) - \theta \cdot w_t\right] = 0,$$

By lemma 4, there exists (x_t, w_{t+1}) in $F_t(g)$ such that

$$w_t = V_t(x_t, w_{t+1}).$$

Also, by optimality,

$$\sum_{i \in J} x_t^i = \sum_{i \in J} e_t^i$$

and

$$\min_{\theta \in \Theta}\left[g_{t+1}(\theta) - \theta \cdot w_{t+1}\right] = 0.$$

By induction, exploiting impatience, along with the fact that the support map is bounded, this procedure defines an allocation x in $X(e)$ such that, at every t in $\mathbb{T}$, for every individual i in J,

$$w_t^i = U_t^i\left(x^i\right),$$

thus proving the claim. □

Proof of proposition 3. Supposing not, at no loss of generality, allocation x in $X(e)$ is *strictly* Pareto dominated by allocation z in $C(e,x) \subset X(e)$. Suppose that, for some t of $\mathbb{T}$,

$$\min_{\theta\in\Theta}\left[g_{t+1}(\theta) - \theta\cdot U_{t+1}(z)\right] \geq 0.$$

Such a period exists by consistency, as consumption adjustments vanish eventually, that is, allocation z lies in $C(e,x)$. This implies that $(z_t, U_{t+1}(z))$ is a feasible element of $F_t(g)$. Hence, by optimality, for every θ in Θ,

$$\begin{aligned} g_t(\theta) &\geq \theta\cdot V_t(z_t, U_{t+1}(z)) \\ &= \theta\cdot U_t(z). \end{aligned}$$

This, by induction, proves that, for all welfare weights θ in Θ,

$$g_0(\theta) \geq \theta\cdot U_0(z) \geq \theta\cdot U_0(x) + \min_{i\in J}\left(U_0^i\left(z^i\right) - U_0^i\left(x^i\right)\right) > \theta\cdot U_0(x).$$

Observing that, by consistency, $g_0(\theta) = \theta\cdot U_0(x)$ for some welfare weights θ in Θ, a contradiction obtains, thus proving the claim. □

Proof of proposition 4. Define g in G, at every t in $\mathbb{T}$, by

$$g_t(\theta) = \sup_{z\in C(e,x)} \theta\cdot U_t(z),$$

where it is understood that this supremum is taken at every state ω in Ω. It is simple to verify that the definition is consistent, that is, g is in fact an element of G. Furthermore, at every t in $\mathbb{T}$,

$$\min_{\theta\in\Theta}\left[g_t(\theta) - \theta\cdot U_t(x)\right] = 0,$$

that is, the Malinvaud constrained efficient allocation x in $X(e)$ maximizes social welfare, for some welfare weights θ in Θ, at every contingency. This is proved in Bloise, Reichlin and Tirelli [8, Lemma 3]. It remains to verify that g in G is indeed a fixed point of the Bellman operator.

Peg any t in $\mathbb{T}$ and notice that z lies in $C(e,x)$ only if $(e_0,\ldots,e_{t-1},z_t,z_{t+1},\ldots)$ is also an element of $C(e,x)$, as participation is satisfied from period t in $\mathbb{T}$ onwards. Observe that

$$g_t(\theta) = \sup_{z\in C(e,x)} \theta\cdot U_t(z) = \sup_{(z_t,y)\in X_t\times C(e,x)} \theta\cdot V(z_t, U_{t+1}(y))$$

subject to

$$\sum_{i\in J} z_t \leq \sum_{i\in J} e_t$$

and

$$V_t^i\left(z_t^i, U_{t+1}^i(y)\right) \geq 0, \quad \forall i\in J.$$

Furthermore, for every y in $C(e,x)$, $g_{t+1}(\theta) \geq \theta\cdot U_{t+1}(y)$ for every θ in Θ. Therefore, $g_t \leq T_t(g_{t+1})$. Suppose that, for some θ in Θ, $g_t(\theta) < T_t(g_{t+1})(\theta)$. (Here, as in the further development of this proof, the construction is conditional on some (non-negligible) event E_t in $\mathcal{F}_t$.) It follows that $\theta\cdot V_t(z_t, w_{t+1}) > g_t(\theta)$ for some feasible plan (z_t, w_{t+1}) in $F_t(g_{t+1})$. Also, by strict monotonicity of utilities with respect to consumptions at t in $\mathbb{T}$, at no loss of generality, $V_t^i\left(z_t^i, w_{t+1}^i\right) > 0$ for every individual i in J. By construction of the map g in G, there exists a sequence of allocations $(y^n)_{n\in\mathbb{N}}$ in $C(e,x)$ such that $U_{t+1}(y^n)$ converges (at least) to w_{t+1} in $W_{t+1}(g_{t+1})$. To verify this, arguing as in the proof of lemma 4, consider the set

$$V_{t+1} = \left\{v_{t+1} \in L_{t+1}\left(\mathbb{R}^J\right) : 0 \leq v_{t+1} \leq U_{t+1}(y) \text{ for some } y \in \bar{C}(e,x)\right\},$$

where $\bar{C}(e,x)$ denotes the topological closure of $C(e,x)$; it is simple to verify that V_{t+1} is a compact convex set in $W_{t+1}(g_{t+1})$; if $(w_{t+1} + L_{t+1}(\mathbb{R}_+)) \cap V_{t+1} = \emptyset$, then there exists θ in $L_{t+1}(\Theta)$ such that, conditional on some (non-negligible) event E_{t+1} in $\mathcal{F}_{t+1}$,

$$g_{t+1}(\theta) \geq \theta \cdot w_{t+1} > \sup_{y \in C(e,x)} \theta \cdot U_{t+1}(y),$$

thus yielding a contradiction. For every sufficiently large n in $\mathbb{N}$, the allocation $z^n = \left(e_0, \ldots, e_{t-1}, z_t, y^n_{t+1}, y^n_{t+2}, \ldots\right)$ lies in $C(e,x)$ and satisfies

$$\theta \cdot U_t(z^n) > g_t(\theta) \geq \sup_{z \in C(e,x)} \theta \cdot U_t(z),$$

thus delivering a contradiction. □

Proof of lemma 5. The proof is simple but laborious for notation, so that I omit obvious details whenever the argument can be trivially expanded. Preliminarily observe that, by the irreducibility of the Markov chain and the finiteness of the Markov state space, $\vec{g}^*(s) = 0$, for some state s in $\bar{S}$, only if $\vec{g}^*(s) = 0$, for every state s in $\bar{S}$. In this latter case, there is nothing to prove. Therefore, it can be assumed that, for some $\epsilon > 0$, at every t in $\mathbb{T}$, $\inf_{\theta\in\Theta} g^*_t(\omega)(\theta) \geq \epsilon$ for every state of nature ω in Ω.

As a simple application of the Principle of Optimality, given any n in $\mathbb{N}$, the operator $T^n : \mathcal{M}(G) \to \mathcal{M}(G)$ corresponds to a maximization program in which, at every t in $\mathbb{T}$, the planner redistributes consumptions $(x_t, \ldots, x_{t+n-1})$ in $X_t \times \cdots \times X_{t+n-1}$ and continuation utility surpluses w_{t+n} in $W_{t+n}(g)$ subject to feasibility. The feasible set, $F^n_t(g)$, requires participation constraints of the form

$$\begin{aligned}
V^1_{t+n-1}(x_{t+n-1}, w_{t+n}) &\geq 0, \\
V^2_{t+n-2}(x_{t+n-2}, x_{t+n-1}, w_{t+n}) &\geq 0, \\
&\vdots \\
V^{n-1}_{t+1}(x_{t+1}, \ldots, x_{t+n-1}, w_{t+n}) &\geq 0, \\
V^n_t(x_t, \ldots, x_{t+n-1}, w_{t+n}) &\geq 0,
\end{aligned}$$

where the aggregator is recursively defined by

$$V^{n+1}_t(x_t, \ldots, x_{t+n}, w_{t+n+1}) = V_t\left(x_t, V^n_{t+1}(x_{t+1}, \ldots, x_{t+n}, w_{t+n+1})\right),$$

beginning with $V^1_t = V_t$.

I shall argue by contradiction. Supposing not, because of the hypotheses of a finite Markov state space, $\bar{S}$, it can be assumed that, at some t in $\mathbb{T}$, conditional on some non-negligible event E_t in $\mathcal{F}_t$ (with respect to which I carry over the argument), there exists θ in Θ such that, for every sufficiently large n in $\mathbb{N}$,

$$\lambda g^*_t(\theta) \geq T^n(\lambda g^*)_t(\theta).$$

(I here exploit the fact that, by monotone concavity, the sequence $(T^n(\lambda g^*))_{n\in\mathbb{N}}$ in $\mathcal{M}(G)$ is weakly increasing.) Consider the constrained efficient allocation x in $X(e)$ satisfying

$$\theta \cdot U_t(x) = g^*_t(\theta).$$

For every n in $\mathbb{N}$, convexity of the feasible set implies

$$\left(e_t + \lambda\left(x_t - e_t\right), \ldots, e_{t+n-1} + \lambda\left(x_{t+n-1} - e_{t+n-1}\right), \lambda U_{t+n}\left(x\right)\right) \in F_t^n\left(\lambda g^*\right).$$

This yields

$$\begin{aligned} \lambda\theta \cdot U_t\left(x\right) &\geq \lambda g_t^*\left(\theta\right) \\ &\geq \theta \cdot V_t^n\left(e_t + \lambda\left(x_t - e_t\right), \ldots, e_{t+n-1} + \lambda\left(x_{t+n-1} - e_{t+n-1}\right), \lambda U_{t+n}\left(x\right)\right). \end{aligned}$$

And, in the limit,

$$\lambda\theta \cdot U_t\left(x\right) \geq \theta \cdot U_t\left(e + \lambda\left(x - e\right)\right) \geq \lambda\epsilon > 0,$$

where strict positivity is implied by the initial remark in this proof. This conflicts with strict concavity of utilities, thus delivering a contradiction. □

Proof of proposition 5. Take any support map g in G satisfying, for some $\lambda > 0$, $g^* > g \geq \lambda g^*$. Define

$$\lambda^* = \sup\left\{\lambda > 0 : g \geq \lambda g^*\right\}.$$

By monotonicity and strict concavity (lemma 5),

$$g = T^n\left(g\right) \geq T^n\left(\lambda^* g^*\right) \geq f\left(\lambda^*\right) g^*,$$

which is a contradiction. To complete the proof, observe that, whenever g lies in $\mathcal{M}\left(G\right)$, because of the indecomposability of the Markov process affecting fundamentals and the fact that autarchy is absorbing, $\vec{g}\left(s\right) = 0$, for some s in $\bar{S}$, only if $\vec{g}\left(s\right) = 0$, for every s in $\bar{S}$. Hence, whenever g in $\mathcal{M}\left(G\right)$ is not autarchic, there certainly exists $\lambda > 0$ such that $g \geq \lambda g^*$. □

Proof of proposition 6. By uniform strict positivity, there exists a sufficiently small $1 > \lambda > 0$ such that

$$\lambda g^* \leq g \leq \frac{1}{\lambda} g^*.$$

Monotonicity and concavity of the Bellman operator imply that $\left(T^n\left(\lambda g^*\right)\right)_{n\in\mathbb{N}}$ in $\mathcal{M}\left(g\right)$ is weakly increasing and that $\left(T^n\left(\left(1/\lambda\right) g^*\right)\right)_{n\in\mathbb{N}}$ in $\mathcal{M}\left(G\right)$ is weakly decreasing. Furthermore, for every n in $\mathbb{N}$,

$$T^n\left(\lambda g^*\right) \leq T^n\left(g\right) \leq T^n\left(\frac{1}{\lambda} g^*\right).$$

I shall show that the dominated increasing orbit converges uniformly to g^* in $\mathcal{M}\left(G\right)$. A similar argument applies to the dominant decreasing orbit, thus proving the claim. By the Markov hypothesis, the relevant space for maps, up to obvious identifications, is $C\left(\Theta\right)^{\bar{S}}$, where $\bar{S}$ is the finite Markov space affecting fundamentals.

Let $f^n = T^n\left(\lambda g^*\right)$ in $\mathcal{M}\left(G^*\right)$ for every n in $\mathbb{N}$. As this sequence in monotonically increasing, it point-wise converges to f in $B\left(\Theta\right)^{\bar{S}}$, where $B\left(\Theta\right)$ denotes the space of bounded real-valued maps on Θ. Peg any arbitrary $\epsilon > 0$ and let Θ^* be a finite subset of Θ satisfying $\min_{\theta^* \in \Theta^*} \left\|\theta - \theta^*\right\| < \left(\epsilon/3\kappa\right)$ for every θ in Θ, where $\kappa > 0$ is given in lemma 2. Hence, for every sufficiently large n in $\mathbb{N}$,

$$\begin{aligned} \left\|f\left(\theta\right) - f^n\left(\theta\right)\right\| &\leq \left\|f\left(\theta\right) - f\left(\theta^*\right)\right\| \\ &\quad + \left\|f\left(\theta^*\right) - f^n\left(\theta^*\right)\right\| \\ &\quad + \left\|f^n\left(\theta^*\right) - f^n\left(\theta\right)\right\| \\ &< \epsilon, \end{aligned}$$

which shows uniform convergence and, hence, exploiting Aliprantis and Border [2, Theorem 2.65], that f is an element of $C(\Theta)^{\bar{S}}$. It is simple to verify that f is a positive convex map satisfying properties (G-1)-(G-2) and, thus, f is an element of $\mathcal{M}(G^*)$. By continuity, it is a fixed point of the Bellman operator, that is, $T(f) = f$. Thus, by the previous proposition about uniqueness, such a fixed point coincides with the constrained efficient support map. □

Proof of lemma 6. Notice that the space of Markov maps $\mathcal{M}(G)$, which is invariant for the Bellman operator, can be identified with a subset of $C\left(\Theta, \mathbb{R}^{\bar{S}}\right)$, endowed with the norm

$$\|f\| = \sup_{\theta\in\Theta} \sup_{s\in\bar{S}} |f_s(\theta)| .$$

Also, the restricted Bellman operator $T : \mathcal{M}(G^*) \to \mathcal{M}(G^*)$ is continuous in the above supremum norm. Notice that the operator is restricted in both domain and range; also, uniform and product topology coincide because of the hypothesis of a finite Markov space $\bar{S}$. I now construct a double sequence indexed by (n, m) in $\mathbb{Z} \times \mathbb{N}$ and then I take the limit along subsequences of m in $\mathbb{N}$.

Given λ_0 in $(0, 1)$, for every m in $\mathbb{N}$, choose λ_m in $(0, 1)$ such that

$$\lambda_0 = \sup\{\lambda \in [0, 1] : T^m(\lambda_m g^*) \geq \lambda g^*\} .$$

For n in $\mathbb{Z}$, define

$$g^{n,m} = T^{(n+m)\vee 0}(\lambda_m g^*),$$

an element of $\mathcal{M}(G^*)$. Notice that, for every n in $\mathbb{Z}$ with $n + m \geq 0$,

$$g^{n+1,m} = T(g^{n,m}) \geq g^{n,m}, \tag{*}$$

where the last inequality is an implication of monotonicity and concavity of the Bellman operator, as

$$T(\lambda_m g^*) \geq \lambda_m g^* \text{ only if } T^{n+1}(\lambda_m g^*) \geq T^n(\lambda_m g^*) .$$

To carry over the proof, I exploit the Ascoli-Arzelà Theorem.

Ascoli-Arzelà Theorem. *A set $\mathcal{K}$ of $C\left(\Theta, \mathbb{R}^{\bar{S}}\right)$ is compact, in the supremum norm, if and only if it is bounded, closed and equicontinuous.*

Consider

$$\mathcal{K} = \text{closure } T(\{g \in \mathcal{M}(G) : g \leq g^*\}) \subset C\left(\Theta, \mathbb{R}^{\bar{S}}\right).$$

Notice that, by lemma 3, $\mathcal{K}$ is contained in $\mathcal{M}(G^*)$. Furthermore, $\mathcal{K}$ is bounded and, by lemma 2, equicontinuous. Therefore, by Ascoli-Arzelà Theorem, is a compact set in $\mathcal{M}(G^*)$.

By compactness, some subsequence $\left(g^{0,m_0(j)}\right)_{j\in\mathbb{N}}$ in $\mathcal{K}$ converges to g^0 in $\mathcal{K} \subset \mathcal{M}(G^*)$. Suppose that, for some n in $\mathbb{Z}_+$, the sequence $\left(g^{n,m_0(j)}\right)_{j\in\mathbb{N}}$ in $\mathcal{K}$ converges to g^n in $\mathcal{K} \subset \mathcal{M}(G^*)$; then, by continuity (lemma 1), the sequence $\left(g^{n+1,m_0(j)}\right)_{j\in\mathbb{N}}$ in $\mathcal{K}$ converges to g^{n+1} in $\mathcal{K} \subset \mathcal{M}(G^*)$, since $g^{n+1,m_0(j)} = T\left(g^{n,m_0(j)}\right)$ for every j in $\mathbb{N}$; in the limit,

$$g^{n+1} = T(g^n) .$$

Suppose that, for some n in $\mathbb{Z}_-$, the subsequence $\left(g^{n,m_n(j)}\right)_{j\in\mathbb{N}}$ in $\mathcal{K}$ converges to g^n in $\mathcal{K} \subset \mathcal{M}(G^*)$; the subsequence $\left(g^{n-1,m_n(j)}\right)_{j\in\mathbb{N}}$ lies in the compact set $\mathcal{K}$ and, hence, some subsequence $\left(g^{n-1,m_{n-1}(j)}\right)_{j\in\mathbb{N}}$ in $\mathcal{K}$ converges to g^{n-1} in $\mathcal{K}\subset\mathcal{M}(G^*)$; furthermore, by continuity (proposition 1),

$$g^n = T\left(g^{n-1}\right).$$

To conclude the proof, observe that condition (*) ensures that, for every n in $\mathbb{Z}$,

$$g^{n+1} = T\left(g^n\right) \geq g^n.$$

Hence, by argument analogous to those in the proof of proposition 6, the weakly decreasing sequence $\left(g^{-n}\right)_{n\in\mathbb{N}}$ of $\mathcal{K}\subset\mathcal{M}(G^*)$ and the weakly increasing sequence $\left(g^n\right)_{n\in\mathbb{N}}$ of $\mathcal{K}\subset\mathcal{M}(G^*)$ converge, respectively, to g^- in $\mathcal{M}(G^*)$ and g^+ in $\mathcal{M}(G^*)$. Observing that $g^-\leq g^0\leq g^+$, this suffices to show that $g^-=0$ and $g^+=g^*$ in $\mathcal{M}(G^*)$, as these are the only two fixed points of the Bellman operator in $\mathcal{M}(G^*)$ (proposition 4). □

Proof of proposition 7. Given any connecting orbit in lemma 6, define g in G, at every t in $\mathbb{T}$, by

$$g_t = g_t^{-t}.$$

Observe that, at every t in $\mathbb{T}$,

$$g_t = g_t^{-t} = T_t\left(g_{t+1}^{-(t+1)}\right) = T_t\left(g_{t+1}\right),$$

thus proving the claim. □

Proof of proposition 8. Given any non-autarchic support map g in $\mathcal{E}(G)$, let the continuous map $\vec{g}: S\to C(\Theta)$ be its reduced representation. Consider the continuous map $\vec{\lambda}: S\to[0,1]$ which is defined by

$$\vec{\lambda}(s) = \sup\left\{\lambda\in[0,1]:\vec{g}(s)\geq\lambda\vec{g}^*(s)\right\},$$

where $\vec{g}^*: S\to C(\Theta)$ represents constrained efficiency. (Remember that this latter map only reflects intrinsic uncertainty and, hence, is continuous.) As both constrained efficiency and autarchy are invariant sets, I can assume that $\vec{\lambda}\left(\mathring{S}\right)\subset(0,1)$. In other terms, both constrained efficiency and autarchy do not appear in $\mathring{S}$. At no loss of generality, I also assume that S coincides with the closure of $\mathring{S}$. For notational convenience, set $S^0=\left\{s\in S:\vec{\lambda}(s)=0\right\}$. The proof requires to establish two auxiliary claims.

Claim 1. *Given any $1>\lambda>0$, for every s in $\mathring{S}$, there exists n in $\mathbb{N}$ such that*

$$\Pi_s^n\left(\left\{s^*\in S:\lambda>\vec{\lambda}(s^*)\right\}\right)>0.$$

Proof. By proposition 6, the sequence $\left(T^n(\lambda g^*)\right)_{n\in\mathbb{N}}$ in $\mathcal{M}(G)$ converges to g^* in $\mathcal{M}(G)$. Furthermore, for every n in $\mathbb{N}$,

$$g^*\geq g\geq T^n(g)\geq T^n(\lambda g^*).$$

Suppose that the claim is violated at some state s in $\mathring{S}$ and consider any event E_t in $\mathcal{F}_t$ at some t in $\mathbb{T}$ such that $\sigma_t(E_t)=s$ in S. By the previous remark, conditional on this event E_t in $\mathcal{F}_t$, $g_t\geq T_t^n(\lambda g^*)_t$ and, therefore, $g_t\geq g_t^*$, a contradiction. □

Notice that, by claim 1, it is straightforward to verify that S^0 is non-empty.

Claim 2. *For every open neighborhood U of S^0, there exists $1 > \lambda_U > 0$ such that*

$$\left\{s^* \in S : \lambda_U > \vec{\lambda}(s^*)\right\} \subset U.$$

Proof. If not, there exits a sequence $(s_n)_{n\in\mathbb{N}}$ in (S/U) such that, for every n in $\mathbb{N}$, $0 < \vec{\lambda}(s_n) < 1/n$. As (S/U) is compact, it can be assumed that the sequence converges to s_∞ in (S/U). However, by continuity, in the limit, $\vec{\lambda}(s_\infty) = 0$ and, hence, s_∞ lies in U, a contradiction. □

Peg any open neighborhood U of S^0. Noticing $\pi\left(\mathring{S}\right) = 1$, by invariance of the probability measure, one obtains

$$\begin{aligned}
\pi(U) &= \sum_{n\in\mathbb{N}} \frac{1}{2^n} \int_S \Pi_s^n(U)\, d\pi(s) \\
&= \sum_{n\in\mathbb{N}} \frac{1}{2^n} \int_{\mathring{S}} \Pi_s^n(U)\, d\pi(s) \\
&= \int_{\mathring{S}} \sum_{n\in\mathbb{N}} \frac{1}{2^n} \Pi_s^n(U)\, d\pi(s) \\
&\geq \int_{\mathring{S}} \sum_{n\in\mathbb{N}} \frac{1}{2^n} \Pi_s^n\left(\left\{s^* \in S : \lambda_U > \vec{\lambda}(s^*)\right\}\right) d\pi(s) \\
&> 0.
\end{aligned}$$

This implies that S^0 intersects the support of the ergodic invariant measure π in $\Delta(S)$. Indeed, if not, as the support and S^0 are both compact sets in S, some open neighborhood U of S^0 has no contact point with the support, contradicting $\pi(U) > 0$.

Observe that S^0 is an invariant set for the Markov transition $\Pi : S \to \Delta(S)$. Indeed, supposing not, there exists s in S^0 such that, for some $1 > \lambda > 0$,

$$\Pi_s\left(\left\{s^* \in S : \vec{\lambda}(s^*) > \lambda\right\}\right) > \gamma > 0.$$

One can peg a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathring{S}$ approaching s in S^0. By the Strong Feller Property, for every sufficiently large n in $\mathbb{N}$,

$$\Pi_{s_n}\left(\left\{s^* \in S : \vec{\lambda}(s^*) > \lambda\right\}\right) > \gamma > 0.$$

This easily contradicts the fact that $\left(\vec{\lambda}(s_n)\right)_{n\in\mathbb{N}}$ in $(0,1)$ is vanishing, as some social surplus can be extracted when the economy remains away from autarchy with some positive probability.

To conclude the argument, observe that S^0 is obviously the support of an invariant measure π^0 in $\Delta(S)$. Furthermore, $\pi\left(S/S^0\right) = \pi^0\left(S^0\right) = 1$. Under the Strong Feller Property, the (topological) supports of two mutually singular invariant probability measures need be disjoint (Hairer [14, Proposition 2.6]). This reveals a contradiction. □

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