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Working Paper n° 13, 2000

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# FRACTIONAL BAYES FACTORS FOR THE ANALYSIS OF AUTOREGRESSIVE MODELS WITH POSSIBLE UNIT ROOTS*

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**Abstract.** *In this paper we consider the problem of identifying an autoregressive model for an observed time series and detecting a possible unit root in its characteristic polynomial. This is a big issue concerned with distinguishing stationary time series from time series for which differencing is required to induce stationarity. We adopt the Bayes approach and assume that the prior information about the parameters of the models is weak. For the comparison of the models in this setting we introduce a modified version of the fractional Bayes factor.*

**Key words:** *Autoregressive model, fractional Bayes factor, model selection, time series, unit root.*

## 1. Introduction

Calculation of a suitable Bayes factor is required for Bayesian model comparison. In recent years, several alternative Bayes factors have been introduced to address the problem of the sensitivity of the usual Bayes factor when prior information is weak. Among these alternatives, the *fractional Bayes factor* (O'Hagan, 1995) and the *intrinsic Bayes factor* (Berger and Pericchi, 1996) perform well for a variety of situations involving sequences of *i.i.d.* data.

The intrinsic Bayes factor approach has been discussed for model selection with dependent data structures in Varshavsky (1996); the purpose of the present paper is to explore the use of the fractional Bayes factor within the same framework. In particular, we deal with the identification of an autoregressive model for a time series and the contemporary test for unit roots, which is still one of the most debated issues in Bayesian time series analysis. The use of fractional Bayes factor for the detection of a change point in the dynamic structure of a time series was already explored in Barbieri and Conigliani (1998).

There are various other published Bayesian analyses which relate directly or indirectly to this problem, with great attention devoted to the nature of suitable noninformative priors for the autoregressive coefficients. Just to mention a few examples: Sims(1988) and Sims and Uhlig (1991) advocate the use of flat priors; Phillips (1991) finds that flat priors bias the inference towards stationary models, and suggests instead the use of Jeffreys priors; for the $AR(1)$ model, Berger and Yang (1994) consider a reference prior approach; Marriott and Newbold (1998) criticise the use of noninformative priors for the autoregressive coefficients in this setting, and suggest the use of (informative) sharp beta prior; Marinucci and Petrella (1999), again only for the $AR(1)$ model, derive a noninformative prior by considering the functional relationship between the autoregressive parameter and the variance of the process. Our method seems, however, to be the first attempt to treat the problem of testing for unit roots as one of comparing a stationary autoregressive model with an integrated model when there is negligible prior information about all the parameters in the models, so that noninformative prior distributions apply not only for the autoregressive coefficients, but also for the mean of the process and the variance of the white noise.

In Section 2 we introduce the models and the notation for the problem of identifying an autoregressive model and testing for a unit root. In Section 3 we introduce Bayes factors and fractional Bayes factors, and we suggest a modification of the fractional Bayes factor for model choice with dependent data. The modified fractional Bayes factor turns out to be a very flexible and powerful tool for tackling various problems that arise in the analysis of dependent data. The particular problem of identifying the order of the model and testing for a unit root is considered in Section 4, with an example on some simulated time series. A few concluding remarks are presented in the final section.

## 2. Models and notation

Let $\{Y_t; t \in \mathcal{Z}\}$ be a Gaussian stationary stochastic process with mean $\mathbb{E}(Y_t) = \mu$ and variance $\gamma(0)$. The process $\{Y_t\}$ is said to be an autoregressive process of order $p$ $(AR(p))$ if it may be represented as

$$\phi(B)(Y_t - \mu) = \epsilon_t$$

where $\phi(B) = 1 - \phi_1 B - ... - \phi_p B^p$, $B$ is the backshift operator, such that $B^k y_t = y_{t-k}$, and $\{\epsilon_t\}$ is a Gaussian white noise process with variance $\sigma^2$. The stationarity condition constrains the parameter vector $\phi^{(p)} = (\phi_1, \phi_2, ..., \phi_p)$ to lie in the stationarity region $\Phi_p = \left\{\phi^{(p)} : \phi(z) = 0 \text{ implies } |z| > 1\right\}$.

Suppose we observe the time series $\mathbf{y} = (y_1, \ldots, y_n)$, a finite realization of an autoregressive process of order $p$, with $p$ unknown; the goal is the identification of $p$, and the contemporary test for a unit root in the autoregressive polynomial.

Recall that if in the generating process there is a number $d$ of unit autoregressive roots, stationarity can be achieved by differencing $d$ times an original series. It follows that for a given order $p$, the test for a single unit root can be performed by comparing the stationary $AR(p)$ model on the original series with the (stationary) autoregressive model of order $p-1$ on the series differenced once, *i.e.* the $ARI(p-1,1)$ model (Box and Jenkins, 1970):

$$\psi(B) z_t = \eta_t,$$

where $\psi(B) = 1 - \psi_1 B - ... - \psi_{p-1} B^{p-1}$, $z_t = y_t - y_{t-1}$, and $\{\eta_t\}$ is a Gaussian white noise process with variance $\tau^2$.

If the order $p$ is also unknown, the problem of identifying the model and testing for a unit root can be seen as the one of choosing a model in the set $\{AR(p),\ ARI(p-1,1); p = 1, 2, ..., p_{\max}\}$. More in general, the identification of the model and the test for un unknown number of unit roots can be seen as the one of choosing a model in the set $\{AR(p), ARI(p-d,d);\ p = 1, 2, ..., p_{\max}; d = 1, 2, ..., d_{\max}\}$; note that this generalisation is straightforward, and will not be considered explicitly in this paper.

### 2.1 About the likelihood

Assume for the moment that the order $p$ is fixed, and consider the problem of comparing the stationary $AR(p)$ model on the original series, denoted with $M_p$, with the (stationary) autoregressive model of order $q = p - 1$ on the series differenced once, *i.e.* the $ARI(q,1)$ model, denoted with $M_q^1$.

Note that because both $M_p$ and $M_q^1$ are stationary models, we can define for both of them the exact likelihood function. Given the first observation $y_1$, the exact likelihood function for $M_q^1$ can be written in term of the differenced time series $\mathbf{z} = (z_2, \ldots, z_n)$:

$$f_q\left(\mathbf{z}\,|\psi^{(q)}, \tau^2, y_1\right) = \left(2\pi\tau^2\right)^{-n/2} |V_q|^{1/2}$$
$$\times \exp\left\{-\frac{1}{2\tau^2}\left[\sum_{t=2}^{q+1}\sum_{s=2}^{q+1} v_{t-1,s-1} z_t z_s\right.\right.$$
$$\left.\left. + \sum_{t=q+2}^{n}\left(z_t - \sum_{h=1}^{q}\psi_t z_{t-h}\right)^2\right]\right\} \tag{1}$$

where $V_q$ is a $q \times q$ matrix whose elements $v_{ij}$ are functions of the parameters $\psi_i$ (Galbraith and Galbraith, 1974).

Similarly, the exact likelihood function for $M_p$ is defined as

$$f_p\left(\mathbf{y}\,|\phi^{(p)}, \mu, \sigma^2\right) = \left(2\pi\sigma^2\right)^{-n/2} |V_p|^{1/2}$$
$$\times \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{t=1}^{p}\sum_{s=1}^{p} v_{ts}\left(y_t - \mu\right)\left(y_s - \mu\right)\right.\right.$$
$$\left.\left. + \sum_{t=p+1}^{n}\left(\left(y_t - \mu\right) - \sum_{h=1}^{p}\phi_h\left(y_{t-h} - \mu\right)\right)^2\right]\right\}. \tag{2}$$

Note, however, that in order to use this likelihood for the comparison between $M_p$ and $M_q^1$, we need to impose the same conditioning to $y_1$ that is in (1); if $g(0)$ is such that $\gamma(0) = \sigma^2 g(0)^{-1}$, we obtain:

$$f_p\left(\mathbf{y}\,|\phi^{(p)}, \mu, \sigma^2, y_1\right) = \left(2\pi\sigma^2\right)^{-n/2} |V_p|^{1/2} g(0)^{-1/2}$$

$$\times \exp\left\{-\frac{1}{2\sigma^2}\left[-g(0)\left(y_1-\mu\right)^2+\sum_{t=1}^{p}\sum_{s=1}^{p} v_{ts}\left(y_t-\mu\right)\left(y_s-\mu\right)\right.\right.$$

$$\left.\left.+\sum_{t=p+1}^{n}\left(\left(y_t-\mu\right)-\sum_{h=1}^{p}\phi_h\left(y_{t-h}-\mu\right)\right)^2\right]\right\}. \tag{3}$$

### 2.2 About the prior distributions

As we noted in the Introduction, great attention in the literature has been devoted to the choice of prior distributions for the autoregressive coefficients. Here, following Sims (1988), for these parameters we assume uniform priors on the stationarity region:

$$\pi_p\left(\phi^{(p)}\right)=\frac{1}{volume\left(\Phi_p\right)} I_{\Phi_p}\left(\phi^{(p)}\right)$$



under $M_p$, and

$$\pi_q\left(\psi^{(q)}\right)=\frac{1}{volume\left(\Psi_q\right)} I_{\Psi_q}\left(\psi^{(q)}\right)$$

under $M_q^1$; these priors were found to behave well also by Berger and Yang (1994).

It is important to note that because of the stationarity constraints, the uniform prior specified above, as well as any other noninformative prior suggested in the literature, are proper, so that the standard Bayesian approach to model selection can be considered for comparing $M_p$ and $M_q^1$. On the other hand, particular care should be devoted to the choice of noninformative prior distributions for the mean of the process, $\mu$, and for the variances $\sigma^2$ and $\tau^2$ of the white noise processes $\{\epsilon_t\}$ and $\{\eta_t\}$. In particular, as we will discuss in the following section, if we adopt standard noninformative prior distributions on these parameters

$$\pi_p\left(\mu,\sigma^2\right)\propto 1/\sigma^2$$

$$\pi_q\left(\tau^2\right)\propto 1/\tau^2$$

we can no longer compute Bayes factors and posterior odds for comparing $M_p$ and $M_q^1$, and alternative tools are needed.

## 3. Bayes factors and fractional Bayes factors

Suppose we are comparing two models, $M_0$ and $M_1$, and let $f_i(\mathbf{y} \mid \boldsymbol{\theta}_i)$ and $\pi_i(\boldsymbol{\theta}_i)$ be respectively the distribution of the data and the prior distribution of the parameters $\boldsymbol{\theta}_i$ under model $M_i$. The Bayes factor for $M_0$ against $M_1$ is defined as

$$B_{01}(\mathbf{y}) = \frac{\int f_0(\mathbf{y} \mid \boldsymbol{\theta}_0)\, \pi_0(\boldsymbol{\theta}_0)\, d\boldsymbol{\theta}_0}{\int f_1(\mathbf{y} \mid \boldsymbol{\theta}_1)\, \pi_1(\boldsymbol{\theta}_1)\, d\boldsymbol{\theta}_1},$$

and it represents the weight of evidence in the data in favour of $M_0$ against $M_1$.

Many problems arise, however, when using the Bayes factor if the prior information is weak, mainly as a consequence of its sensitivity to prior assumptions. In fact, when the prior distributions are proper but diffuse, the more flat the prior, the more penalised the corresponding model; moreover, when they are improper, *i.e.* defined only up to arbitrary constants, then the Bayes factor is itself a multiple of these constants.

Thus, when the prior information is weak, various authors suggest the use of partial Bayes factors: the idea is to use part of the data as a training sample to update the prior distributions, and the remainder of the data to compare the models. Formally, divide the data into two parts, $\mathbf{y} = (\mathbf{x}, \mathbf{w})$, of size $m$ and $n - m$ respectively, with $0 < m < n$. First, subsample $\mathbf{x}$ is used to obtain the posterior distributions $\pi_i(\boldsymbol{\theta}_i \mid \mathbf{x})$; in the second step, taking these as prior distributions, the remaining data $\mathbf{w}$ are used to compute a Bayes factor

$$\begin{aligned} B_{01}(\mathbf{w} \mid \mathbf{x}) &= \frac{\int f_0(\mathbf{w} \mid \boldsymbol{\theta}_0)\, \pi_0(\boldsymbol{\theta}_0 \mid \mathbf{x})\, d\boldsymbol{\theta}_0}{\int f_1(\mathbf{w} \mid \boldsymbol{\theta}_1)\, \pi_1(\boldsymbol{\theta}_1 \mid \mathbf{x})\, d\boldsymbol{\theta}_1} \\ &= \frac{\int f_0(\mathbf{y} \mid \boldsymbol{\theta}_0)\, \pi_0(\boldsymbol{\theta}_0)\, d\boldsymbol{\theta}_0}{\int f_0(\mathbf{x} \mid \boldsymbol{\theta}_0)\, \pi_0(\boldsymbol{\theta}_0)\, d\boldsymbol{\theta}_0} \Big/ \frac{\int f_1(\mathbf{y} \mid \boldsymbol{\theta}_1)\, \pi_1(\boldsymbol{\theta}_1)\, d\boldsymbol{\theta}_1}{\int f_1(\mathbf{x} \mid \boldsymbol{\theta}_1)\, \pi_1(\boldsymbol{\theta}_1)\, d\boldsymbol{\theta}_1}. \end{aligned} \tag{4}$$

As pointed out in O'Hagan (1995), partial Bayes factors are less sensitive to prior distributions than Bayes factors; furthermore, partial Bayes factors do not depend on arbitrary constants when improper priors are used. There is, however, a difficulty with the use of partial Bayes factors, namely the selection of the training sample $\mathbf{x}$ from the data.

To avoid the arbitrariness of choosing a particular training sample or having to consider all possible subsets of a given size, O'Hagan (1991) suggested instead the use of a proportion $b$ of the data for training. Formally, let $b = m/n$; if both $m$ and $n$ are large, the likelihood $f_i\left(\mathbf{x} \mid \boldsymbol{\theta}_i\right)$ based only on the training sample $\mathbf{x}$ will approximate to the full likelihood $f_i\left(\mathbf{y} \mid \boldsymbol{\theta}_i\right)$ raised to the power of $b$:

$$f_i\left(\mathbf{x} \mid \boldsymbol{\theta}_i\right) \approx f_i\left(\mathbf{y} \mid \boldsymbol{\theta}_i\right)^b . \tag{5}$$

By analogy with (4), this motivates the definition of the fractional Bayes factor (O'Hagan, 1995):

$$B_{01}^b\left(\mathbf{y}\right) = \frac{\int f_0\left(\mathbf{y} \mid \boldsymbol{\theta}_0\right) \pi_0\left(\boldsymbol{\theta}_0\right) d\boldsymbol{\theta}_0}{\int f_0\left(\mathbf{y} \mid \boldsymbol{\theta}_0\right)^b \pi_0\left(\boldsymbol{\theta}_0\right) d\boldsymbol{\theta}_0} \Big/ \frac{\int f_1\left(\mathbf{y} \mid \boldsymbol{\theta}_1\right) \pi_1\left(\boldsymbol{\theta}_1\right) d\boldsymbol{\theta}_1}{\int f_1\left(\mathbf{y} \mid \boldsymbol{\theta}_1\right)^b \pi_1\left(\boldsymbol{\theta}_1\right) d\boldsymbol{\theta}_1} .$$

Among the several alternatives introduced in recent years to address the problem of the sensitivity of the usual Bayes factor to prior assumptions, the fractional Bayes factor makes an important contribution on the ground of consistency, coherence and robustness, and perform well for a variety of situations involving sequences of *i.i.d.* data (O'Hagan, 1995, 1997).

### 3.1 A modification of fractional Bayes factor for autoregressive data

Consider again the situation where $\mathbf{y}$ is a realization of an autoregressive process, and for the moment consider the simple problem of identifying the order of the autoregressive model. More specifically, suppose that we are comparing an autoregressive model of order $p$, $M_p$, with an autoregressive model of order $q$, $M_q$, with fixed $p$ and $q$. To simplify the notation, let $\boldsymbol{\theta}_p$ and $\boldsymbol{\theta}_q$ be the vectors of parameters under $M_p$ and $M_q$ respectively, with noninformative prior distributions $\pi_p(\boldsymbol{\theta}_p)$ and $\pi_q(\boldsymbol{\theta}_q)$ defined as in Section 2.2. Note that in this simplified problem, unlike in the setting described in Section 2, for both models we consider the exact likelihood function (2).

Now, for this particular problem, we can show that (5) is not true, but that the following approximation holds:

$$f_r(\mathbf{x} \mid \boldsymbol{\theta}_r) \approx f_r(\mathbf{y} \mid \boldsymbol{\theta}_r)^b |V_r|^{(1-b)/2} \qquad r = p, q. \tag{6}$$

We can then define a modified version of the fractional Bayes factor for $M_p$ against $M_q$:

$$B^b_{pq}(\mathbf{y}) = \frac{\int f_p(\mathbf{y}|\boldsymbol{\theta}_p)\,\pi_p(\boldsymbol{\theta}_p)\,d\boldsymbol{\theta}_p}{\int f_p(\mathbf{y}|\boldsymbol{\theta}_p)^b C^F \pi_p(\boldsymbol{\theta}_p)\,d\boldsymbol{\theta}_p} \Big/ \frac{\int f_q(\mathbf{y}|\boldsymbol{\theta}_q)\,\pi_q(\boldsymbol{\theta}_q)\,d\boldsymbol{\theta}_q}{\int f_q(\mathbf{y}|\boldsymbol{\theta}_q)^b C^F \pi_q(\boldsymbol{\theta}_q)\,d\boldsymbol{\theta}_q} \tag{7}$$

where $C^F = |V_r|^{(1-b)/2}$ is the correction factor induced by (6), that only depends on the autoregressive coefficients $\boldsymbol{\phi}^{(r)}$ $(r = p, q)$. Note that (7) can be used to compare $M_p$ and $M_q$, or to address any other problem of model comparison in the autoregressive framework, such as the detection of a change point, for which the likelihood is defined as in (2).

Now consider again the problem introduced in Section 2, namely the problem of testing for a unit root. As we have already discussed, since in (1) there is a conditioning to the first observation $y_1$, we imposed the same conditioning also in the likelihood under $M_p$, and considered (3) instead of (2). It follows that for this specific problem, it is necessary to find a new approximation to replace (5), that takes into account the conditioning to $y_1$. In fact, we can show that in this case

$$f_r(\mathbf{x} \mid \boldsymbol{\theta}_r) \approx f_r(\mathbf{y} \mid \boldsymbol{\theta}_r)^b |V_r|^{(1-b)/2} g(0)^{-(1-b)/2} \qquad r = p, q. \tag{8}$$

Now taking into account the correction factor $\widetilde{C}^F = |V_r|^{(1-b)/2} g(0)^{-(1-b)/2}$, as in (7), we can define a modified version of the fractional Bayes factor and use it to compare $M_p$ and $M^1_q$:

$$B^b_{pq}(\mathbf{y}) = \frac{\int f_p(\mathbf{y}|\boldsymbol{\theta}_p)\,\pi_p(\boldsymbol{\theta}_p)\,d\boldsymbol{\theta}_p}{\int f_p(\mathbf{y}|\boldsymbol{\theta}_p)^b \widetilde{C}^F \pi_p(\boldsymbol{\theta}_p)\,d\boldsymbol{\theta}_p} \Big/ \frac{\int f_q(\mathbf{y}|\boldsymbol{\theta}_q)\,\pi_q(\boldsymbol{\theta}_q)\,d\boldsymbol{\theta}_q}{\int f_q(\mathbf{y}|\boldsymbol{\theta}_q)^b \widetilde{C}^F \pi_q(\boldsymbol{\theta}_q)\,d\boldsymbol{\theta}_q}. \tag{9}$$

# 4. A procedure for identifying an autoregressive model and testing for a unit root

We can now propose a procedure for identifying an autoregressive model and testing for a unit root in its characteristic polynomial; as we discussed in Section 2, this problem can be seen as the one of choosing a model in the set $\{AR(p), \ \ ARI(p-1,1); p=1,2,...,p_{\max}\}$.

For each possible order $p$ we can compare the autoregressive model of order $p$, $M_p$, with the corresponding integrated model of order $q=p-1$, $M_q^1$, by computing the modified fractional Bayes factor (9); let $\widetilde{M}(p)$ be the preferred model. We are then left to choose a model in the set $\left\{\widetilde{M}(p); p=1,2,...,p_{\max}\right\}$, again by computing (9) for all the possible pairs of models.

## 4.1 Computing marginal densities and fractional Bayes factors

Consider first the computation of $\int f_p(\mathbf{y} \mid \boldsymbol{\theta}_p)\, \pi_p(\boldsymbol{\theta}_p)\, d\boldsymbol{\theta}_p$ and $\int f_p(\mathbf{y} \mid \boldsymbol{\theta}_p)^b\, \widetilde{C}^F \pi_p(\boldsymbol{\theta}_p)\, d\boldsymbol{\theta}_p$. Recall that $\boldsymbol{\theta}_p = \left(\mu, \sigma^2, \boldsymbol{\phi}^{(p)}\right)$, and that $\widetilde{C}^F$ only depends on $\boldsymbol{\phi}^{(p)}$. Let $\alpha = (1-\phi_1-...-\phi_p)$, $e_t = y_t - \phi_1 y_{t-1} - ... - \phi_p y_{t-p}$, and

$$w_{ts} = \begin{cases} v_{ts} & (t,s) \neq (1,1) \\ v_{11} - g(0) & (t,s) = (1,1) \end{cases}.$$

Integrating out $\mu$ and $\sigma^2$, for $p=1,2,...,p_{\max}$, we get

$$\int f_p(\mathbf{y} \mid \boldsymbol{\theta}_p)\, \pi_p(\boldsymbol{\theta}_p)\, d\boldsymbol{\theta}_p = \pi^{-(n-2)/2} \Gamma\left(\frac{n-2}{2}\right) \frac{1}{volume(\Phi_p)} \times \int |V_p|^{1/2} g(0)^{-1/2} \zeta^{-1/2} R^{-(n-2)/2} d\boldsymbol{\phi}^{(p)}$$

and

$$\int f_p(\mathbf{y} \mid \boldsymbol{\theta}_p)^b\, \widetilde{C}^F \pi_p(\boldsymbol{\theta}_p)\, d\boldsymbol{\theta}_p = \pi^{-[b(n-1)-1]/2} b^{-b(n-1)/2} \times \frac{\Gamma\left(\frac{b(n-1)-1}{2}\right)}{volume(\Phi_p)} \int |V_p|^{1/2} g(0)^{-1/2} \zeta^{-1/2} R^{-[b(n-1)-1]/2} d\boldsymbol{\phi}^{(p)}$$

where

$$\zeta = \sum_{s=1}^{p} \sum_{t=1}^{p} w_{ts} + \alpha^2 (n-p)$$

$$\widetilde{\mu} = \zeta^{-1} \left( \sum_{s=1}^{p} \sum_{t=1}^{p} w_{ts} y_t + \alpha \sum_{t=p+1}^{n} e_t \right)$$

$$R = \sum_{s=1}^{p} \sum_{t=1}^{p} w_{ts} y_t y_s + \sum_{t=p+1}^{n} e_t^2 - \zeta \widetilde{\mu}^2.$$

Similarly, integrating out $\tau^2$, for $q = 1, 2, ..., (p_{\max} - 1)$, we obtain

$$\int f_q (\mathbf{y} \mid \boldsymbol{\theta}_q) \pi_q (\boldsymbol{\theta}_q) d\boldsymbol{\theta}_q = \pi^{-(n-1)/2} \Gamma \left( \frac{n-1}{2} \right) \frac{1}{volume (\Psi_q)}$$
$$\times \int |V_q|^{1/2} \left[ \sum_{s=1}^{q} \sum_{t=1}^{q} v_{ts} z_t z_s + \sum_{t=q+1}^{n-1} \eta_t^2 \right]^{-(n-1)/2} d\boldsymbol{\psi}^{(q)}$$

and

$$\int f_q (\mathbf{y} \mid \boldsymbol{\theta}_q)^b \widetilde{C}^F \pi_q (\boldsymbol{\theta}_q) d\boldsymbol{\theta}_q = \pi^{-b(n-1)/2} b^{-b(n-1)/2} \Gamma \left( \frac{b(n-1)}{2} \right)$$
$$\times \frac{1}{volume (\Psi_q)} \int |V_q|^{1/2} \left[ \sum_{s=1}^{q} \sum_{t=1}^{q} v_{ts} z_t z_s + \sum_{t=q+1}^{n-1} \eta_t^2 \right]^{-b(n-1)/2} d\boldsymbol{\psi}^{(q)},$$

while when $q = 0$

$$\int f_q (\mathbf{y} \mid \boldsymbol{\theta}_q) \pi_q (\boldsymbol{\theta}_q) d\boldsymbol{\theta}_q = \pi^{-(n-1)/2} \Gamma \left( \frac{n-1}{2} \right) \left( \sum_{t=1}^{n-1} z_t^2 \right)^{-(n-1)/2}$$

and

$$\int f_q (\mathbf{y} \mid \boldsymbol{\theta}_q)^b \widetilde{C}^F \pi_q (\boldsymbol{\theta}_q) d\boldsymbol{\theta}_q = \pi^{-b(n-1)/2} b^{-b(n-1)/2} \Gamma \left( \frac{b(n-1)}{2} \right)$$
$$\times \left( \sum_{t=1}^{n-1} z_t^2 \right)^{-b(n-1)/2}.$$

Note that since it is not possible to obtain analytically the integrals required for the computation of (9), except for the case $q = 0$, it is necessary to resort to some numerical procedures. In particular, we could apply importance sampling, using a large number of random vectors on $\Phi_p$ (or $\Psi_q$) drawn from an appropriate importance function. When $p > 2$, however, even if the volume of the stationarity region is known (Piccolo, 1982), it is difficult to identify its form, so that simulation over this region is not feasible. Instead, we find convenient to reparametrize each $AR(p)$ model in terms of partial autocorrelations $\rho_1, ..., \rho_p$ (Barndorff-Nielsen and Shou, 1973). More specifically, for $k = 1, ..., p$ we introduce $\boldsymbol{\xi}_k = (\xi_{1,k}, \xi_{2,k}, ..., \xi_{k,k})$ defined as

$$\begin{array}{ll} \xi_{1,1} = \rho_1 & \\ \xi_{i,k} = \xi_{i,k-1} - \rho_k \xi_{k-i,k-1} & i = 1, ..., k-1 \\ \xi_{k,k} = \rho_k & k = 2, ..., p \end{array}$$

so that $\boldsymbol{\phi}^{(p)} = \boldsymbol{\xi}_p$. Note that the stationarity constrainsts on the autoregressive coefficients imply $|\rho_k| < 1$ $(k = 1, ..., p)$.

### 4.2 Examples

We simulated 100 data from three different models; as it is shown in the following tables, for all of them the fractional Bayes factor (9) was able to identify the right generating process. In all the examples we let $p_{\max} = 2$, and the importance function was taken to be normal with mean equal to the Durbin estimate of partial autocorrelation.

**Table 1: 100 values simulated from an $AR(1)$ model with $\mu = 0$, $\sigma^2 = 1$, $\phi = 0.23$.**

| |
|---|
| $B^b(AR(1), ARI(0,1)) = 3.06e+007$ |
| $B^b(AR(2), ARI(1,1)) = 4.01e+007$ |
| $B^b(AR(1), AR(2)) = 1.264$ |

**Table 2: 100 values simulated from an $ARI\,(0,1)$ model with $\sigma^2 = 1$.**

| |
|---|
| $B^b\,(AR(1), ARI\,(0,1)) = 0.18$ |
| $B^b\,(AR(2), ARI\,(1,1)) = 5.17e + 007$ |
| $B^b\,(AR\,(2)\,, ARI\,(0,1)) = 0.13$ |

**Table 3: 100 values simulated from an $AR\,(1)$ model with $\mu = 0$, $\sigma^2 = 1$, $\phi = 0.95$.**

| |
|---|
| $B^b\,(AR(1), ARI\,(0,1)) = 1.09$ |
| $B^b\,(AR(2), ARI\,(1,1)) = 1.07e + 008$ |
| $B^b\,(AR(1), AR\,(2)) = 1.33$ |

## 5. Concluding remarks

The results presented here demonstrate the feasibility of our procedure for identifying an autoregressive model and testing for a unit root in its characteristic polynomial; applications on real data sets are currently under investigation. As we mentioned in Section 2, the method can be easily generalised for testing for an unknown number of unit roots; further generalizations, that will be considered elsewhere, include testing for the rise or the loss of a unit root from the characteristic polynomial at a (possibly unknown) specific time.

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