Dipartimento
di Economia
Working Papers

n. 290
2026

Luca Spinesi

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ISSN 2279-6916 Working Papers

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Università degli Studi Roma Tre
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How innovation and heterogenous human capital shape wealth and income inequality

Luca Spinesi
Department of Economics, University of Rome Tre
Via Silvio d'Amico, 77 00145 Rome

March 4, 2026

Abstract

This paper studies how interactions between heterogeneous households in terms of shares of wealth owned, shares of human capital and shares of consumption influence wealth and income inequality in an innovation driven growth model with endogenous market structure. Innovation and workforce skill updating jointly determine long-run economic growth, while patent policy generates non-monotonic distributional effects. Moderate increases in patent breadth raise firms' valuations and dampens wealth and income inequality, whereas strong patent protection increases wealth and income inequality. In contrast, greater time devoted to skill updating unambiguously fosters innovation and long-run growth, but also increases wealth and income inequality. The model is calibrated to U.S. data and shown to replicate key features of the joint evolution of wealth and income inequality over the period 1989–2019.

Keywords: Innovation; Endogenous market structure; Human capital; Income and Wealth.

JEL Classification: O30; O40; D43; J24.

Statements and Declarations: All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

1 Introduction

Innovation is widely regarded as a fundamental engine of long-run economic growth, yet its distributional consequences remain the subject of active debate. A growing empirical literature documents a close association between innovation-driven growth and rising inequality, particularly at the top of the income distribution. Using detailed U.S. data, Aghion et al. (2019) show that innovation-intensive sectors account for a substantial share of the increase in top income inequality in recent decades. Central to this evidence is the strong correlation between patenting activity and innovative output, suggesting that patent-based measures capture an important dimension of technological progress.

Building on this insight, recent theoretical and empirical analyses has examined how patent policy shapes both innovation incentives and the distribution of income. In an influential contribution, Chu et al. (2021) study the effects of patent protection in a growth model with endogenous market structure. Their analysis highlights a fundamental trade-off: stronger patent protection can stimulate innovation and growth in the short run, but it may also exacerbate inequality or generate non-monotonic distributional effects when product variety is fixed. Once firm entry and the expansion of product varieties are allowed for, however, both growth and inequality tend to decline in the long run. When calibrated to U.S. data, their model implies that the long-run inequality-reducing effects of patent policy can outweigh its short-run distributional costs. This result remains consistent with the empirical finding from a panel vector autoregression. Despite these rich transitional dynamics at the aggregate level, existing models often deliver sharp distributional predictions along the transition path. In particular, Chu et al. (2021) show that wealth inequality remains stationary over time, reflecting equilibrium restrictions that pin down consumption-output and consumption-wealth ratios. Recent evidence highlights a sharp and persistent rise in US wealth inequality. Between 1983 and 2016, the top income tier increased its share of national wealth from 60% to 79%, while middle-income families saw their share fall by nearly half, from 32% to 17%. The bottom tier held only 4% of wealth by 2016 (Horowitz et al., 2020). This sustained divergence underscores the need to shed further lights to understand the mechanisms driving long-run wealth concentration.

This manuscript revisits these insights by addressing two empirically relevant issues. First, we take account of the sharp and persistent rise in US wealth inequality. Second, a large empirical literature demonstrates that the growth impact of R&D depends critically on the ability of workers to update their human capital in response to technological change. Classic firm-level studies such as Jaffe (1986) and Cohen and Levinthal (1989) demonstrate that the ability to internalize and exploit new knowledge - often described as absorptive capacity - depends critically on sustained investment in skills. At the cross-country level, Griffith, Redding, and Van Reenen (2003) further show that human-capital-mediated absorptive capacity is central to explaining persistent productivity differences across OECD industries. Griffith, Redding, and Van Reenen (2004) show that human capital not only complements domestic R&D in driving innovation, but also strengthens countries' capacity to absorb frontier technologies, thereby magnifying long-run TFP growth. Consistently, Kwark and Shyn (2006) use data for more than one hundred countries to document that human capital is the primary conduit through which foreign R&D spillovers translate into productivity improvements, underscoring that knowledge diffusion requires continuous skill renewal rather than static educational attainment. At the firm level, the role of updating becomes even more explicit. Dostie (2018) quantifies how training hours devoted to skill upgrading substantially raise firm-level productivity in Canadian manufacturing, while González, Miles-Touya, and Pazo (2016) find that targeted employee training programs significantly increase innovation outcomes among Spanish firms. Recent micro-evidence from Guo, Ning, and Chen (2022) shows that the ability of firms to benefit from foreign technology inflows hinges on maintaining an optimally diverse and continuously refreshed human-capital base. Taken together, these empirical findings reveal a consistent pattern: human capital matters, but its growth-enhancing force operates through active, continuous updating, especially for workers engaged in R&D and innovation-related tasks. This motivates our framework, which considers human-capital updating as an ingredient of the innovation process, thereby linking R&D incentives, firm dynamics, and the long-run distribution of income and wealth.

This study builds on the framework of Chu et al. (2021) and extends it by explicitly incorporating heterogeneous human capital accumulation and its updating as a prerequisite for effective R&D activity. Since R&D is conducted by a human-capital-embodied workforce, individuals must continuously update their skills to remain technologically compatible with new technological advancements. Calibrated to the US economy, the model reproduces the evolution of wealth and income Gini coefficients over recent decades.

The results show a non-monotonic effect of larger patent breadth on wealth and income inequality. Despite these distributional dynamics, the model predicts that long-run economic growth remains stable. Growth is primarily driven by human capital accumulation, which is not directly affected by patent breadth. In particular, an expansion in patent breadth that raises the price of the intermediate variety, reduces firm size and market share. This creates room for entry of new patented varieties. This endogenous adjustment in market structure generates opposing effects on the demand of labor for R&D activity. Smaller firm-level market shares reduce the demand for workers within each incumbent firm, while entry of new firms increases labor demand. Through these channels, patent breadth exerts offsetting effects on the wage flow per unit of human capital. These countervailing forces give rise to a non-monotonic relationship between patent breadth and the expected stock market value of patented products. When patent protection is strengthened only mildly, the positive effect of lower R&D labor costs dominates the negative effect of reduced market shares, leading to an increase in firms' expected stock market values. By contrast, when patent breadth is strengthened further, the negative effect of lower market shares dominates the positive effect of lower R&D labor cost, and the firms' market valuation decline.

This non-monotonic effect of patent breadth on firms' asset valuation also has non-monotonic effect on wealth and income inequality. A central mechanism governing the time evolution of wealth inequality operates through the relative importance of labor income and asset income in households' wealth dynamics. A moderate strengthening of patent protection, by increasing aggregate asset values, attenuates the impact of labor income relative to assets on wealth accumulation, and this gives an advantage to households with a relatively high share of wealth, i.e., the wealthier households. This interaction rises wealth inequality in the short run. However, as time elapses, human capital accumulation allows the poorer households to increase their share of wealth, which is further spurred by the high asset value. This generates a more moderate increase of wealth inequality over a longer time horizon.

By contrast, when patent protection is tightened substantially, the mechanisms described above imply that the resulting decline in aggregate asset values decreases wealth inequality, at least initially. However, as time elapses, human capital accumulation allows the poorer households to increase their share of wealth, yet wealth inequality rises because the very low value of assets disproportionally advantage households with a high enough share of wealth. Over time, wealth inequality increases in both cases. However, it remains persistently lower under a moderate increase in patent breadth, reflecting the higher absolute asset values generated by a less aggressive strengthening of patent protection which tend to allow high enough wealth accumulation of the poorest households, that is not possible when the asset value decreases in the presence of a low share of wealth as happens in the case of stronger patent protection.

Income inequality responds similarly to changes in patent protection. Households with greater asset holdings experience a relatively higher income when the asset value increases due to a moderate increase in patent protection, and this positive effect is further reinforced when the household has a sufficiently large share of human capital. The combination of these two elements implies that a moderate increase in patent breadth disproportionately increases income among wealthier households. By contrast, when patent protection is tightened substantially, the mechanisms described imply that the resulting decline in aggregate asset values mitigates income inequality, at least initially. As time elapses, human capital accumulation together with a more sustained asset accumulation of the richer households increases income inequality. However, as for wealth inequality, income inequality remains persistently lower under a moderate increase in patent breadth reflecting the effect of higher absolute asset values generated by a less aggressive strengthening of patent protection which allow high enough wealth accumulation of the poorer households.

These results extend to an environment with heterogeneous patent breadth across sectors and industries. This robustness is empirically relevant for at least two reasons. First, as a matter of legal practice, patent breadth is often determined through court decisions that adjudicate disputes over specific patents. As a consequence, changes in effective protection are typically confined to particular technologies or industries rather than applied uniformly across the economy (e.g., Scotchmer, 2004). Heterogeneity in patent breadth is therefore a natural feature of real-world innovation systems. Second, a growing body of evidence shows that the impact of patent protection on research investment varies substantially across industries and technological fields (e.g., Roin, 2014; Williams, 2017). This heterogeneity suggests that a uniform patent policy may be suboptimal relative to a more tailored approach that reflects sector-specific innovation environments.

Our framework accommodates this heterogeneity in a tractable way and allows for a systematic analysis of how sector-specific patent protection shapes aggregate innovation, endogenous market structure, and distributional outcomes.

The paper also examines how investments in skill updating interact with innovation, market structure, and inequality. Increasing the time devoted to human capital updating alongside ongoing technological progress accelerates skill accumulation and promotes the expansion of intermediate varieties. These effects raise labor demand and wages, inducing firms to scale up production. When profit margins are sufficiently high and R&D operations are not excessively large, the combined effect of greater firm size and higher R&D labor costs increases firms' market valuations. Both conditions become more easily satisfied in the presence of stronger technological spillovers, which magnify the interaction between market structure and innovation incentives. Higher firm valuations, together with entry and variety expansion, increase aggregate asset values. The distributional consequences mirror those associated with stronger patent protection. Greater updating effort raises wages, human capital growth, and asset prices - components that are initially concentrated among wealthier households - thereby widening inequality in the short run. Over time, however, human capital dynamics generate convergence. Households with initially low human capital shares benefit disproportionately from skill upgrading and experience faster proportional growth, which tempers disparities in both wealth and income. As a result, inequality increases only modestly in the long run.

Because per capita growth is driven by innovation and human capital accumulation - both reinforced by skill updating -greater updating effort raises the long-run growth rate while mitigating the long-run rise in inequality. The model therefore implies a negative relationship between wealth and income inequality and GDP per capita growth in the long run.

The rest of the paper is organized as follows. Section 2 ties the manuscript with the existing related literature. Section 3 sets up the model. Section 4 describes the equilibrium. In Section 5, the dynamics of the variables of interest for the US economy are obtained. Section 6 extends results, while Section 7 draws some conclusions.

2 Related literature

This paper builds on the R&D-based growth literature initiated by Romer (1990), where long-run growth arises from purposeful innovation undertaken by agents with homogeneous human capital. The Schumpeterian tradition - developed by Aghion and Howitt (1992), Grossman and Helpman (1991), and Segerstrom and Dinopoulos (1990) - replaces product variety expansion with quality-improving innovation as the engine of growth. Second-generation models (e.g., Smulders and van de Klundert 1995; Peretto 1998, 1999; Howitt 1999) integrate vertical and horizontal innovation and introduce richer market structures. A growing strand of this literature studies patent policy and its implications for innovation and inequality. Chu (2010) analyzes patent length, Chu (2010b), Chu and Cozzi (2018), and Kiedaisch (2020) analyze how patent strength affects growth and income distribution, but typically within frameworks featuring exogenous market structure. Moreover, several contributions abstract from human capital accumulation and population growth. Chu et al. (2019) analyze monetary policy in a Schumpeterian growth model featuring endogenous market structure and heterogeneous household asset holdings. Chu et al. (2021) combine cumulative innovation and endogenous market structure with heterogeneous asset holdings of households, and examine the inequality-growth nexus under alternative patent regimes. Since the literature on patent design is very rich, for brevity the reader is referred to Chu et al. (2013), Chu and Cozzi (2018), and Chu et al. (2021) for a more detailed analysis of the literature on these aspects.

Our contribution extends this line of research along two dimensions. First, we introduce heterogeneous human capital accumulation, which generates a non-stationary wealth distribution and alters the transmission of patent policy to inequality. Second, we show that the results are robust to heterogeneous patent breadth across sectors, allowing patent protection to vary across industries. This extension is particularly relevant given the extensive literature on patent design (see, among others, Chu et al. 2013; Chu and Cozzi 2018; Chu et al. 2021), and the empirical observation that patent strength is rarely uniform across technologies. More broadly, the paper contributes to the debate on whether patent protection fosters innovation and inclusive growth. While patents are widely used, empirical evidence on their effectiveness remains mixed. Critics such as Boldrin and Levine (2008, 2013), Jaffe and Lerner (2004), and Bessen and Meurer (2008) question the innovation-enhancing role of patents. At the same time, more recent work (e.g., Budish, Roin, and Williams 2015; Moscona 2020) emphasizes that innovation responses depend critically on patent design and sectoral characteristics. Overall, the empirical record remains suggestive rather than definitive (see Sampat and Williams 2019; Bloom et al. 2019; Williams 2013, 2017). By embedding heterogeneous patent breadth and endogenous human capital into a unified growth framework, our model provides a structured approach to analyze these issues.

Finally, the paper relates to the broader literature on inequality, innovation, and growth. This literature is vast, and any brief overview inevitably omits many important contributions. Early cross-country studies (Barro 2000; Forbes 2000; Banerjee and Duflo 2003) document complex relationships between inequality and aggregate growth. Focusing on the upper tail, Frank (2009) finds a positive association between growth and top income shares in the United States. Within R&D-based growth models, several studies explore how innovation shapes income distribution (e.g., Chou and Talmain 1996; Zweimüller 2000; Foellmi and Zweimüller 2006; Aghion et al. 2019). Jones and Kim (2018), in particular, develop a Schumpeterian model linking top income inequality to entrepreneurial dynamics. They show that forces increasing the effort or productivity of high-growth entrepreneurs amplify top inequality, whereas stronger creative destruction reduces it by limiting the persistence of leading innovators.[1]

3 The model

The paper develops a Schumpeterian growth model with in-house R&D and endogenous market structure, building on the creative accumulation framework of Peretto (2007, 2011) and extending Chu et al. (2021). In contrast to creative-destruction models, innovation occurs through the expansion and quality improvement of existing intermediate varieties. This mechanism is consistent with recent empirical evidence showing that creative accumulation, rather than business-stealing entry, accounts for the bulk of technological progress (e.g., Garcia-Macia et al., 2019). We extend the analysis of Chu et al. (2021) considering heterogenous human capital accumulation of agents who allocate their time between R&D activity and skill updating, which affect the improvement of existing varieties and the expansion of varieties in the economy.

For expositional clarity, the baseline specification assumes homogeneous patent breadth across varieties. However, the framework and its main results extend to an environment with heterogeneous patent protection across industries. The formal derivations establishing this robustness are provided in the Appendixes. In addition, the model can be extended to incorporate population growth without altering the core qualitative mechanisms and results.

[1] The analysis of Jones and Kim (2018) considers entrepreneurial income that includes wage, salary and "business income" (i.e., profits from sole proprietorships, partnerships, and S corporations).

3.1 Households

There is a unit continuum of identical households $\kappa \in [0,1]$ that derive utility from consumption. Population is assumed to be constant, and the intergenerational utility of households is the discounted sum of per capita utility across time.[2] The utility function of a household is given by

$$U(\kappa) = \int_0^\infty e^{-\rho t} \ln \left(c_t(\kappa)\right) dt, \tag{1}$$

where the parameter $\rho > 0$ is the discount rate and $c_t(\kappa)$ is the per capita consumption of final goods and services (numeraire). Each household maximizes eq. (1) subject to the following asset accumulation equation

$$\dot{a}_t(\kappa) = r_t a_t(\kappa) + w_t (1-\tau) l_t(\kappa) + \psi(\kappa) \tau w_t L_t - c_t(\kappa), \tag{2}$$

where $a_t(\kappa)$ is the amount of financial assets per capita, r_t is the rate of return on assets, w_t is the wage flow per unit of human capital, and $l_t(\kappa)$ is human capital-embodied labor supply, τ is the payroll tax rate to finance social security projects, $\psi(\kappa)$ is the household's κ share of the social security aggregate entries, $\tau w_t L$, with the constraint $\int_0^1 \psi(\kappa) d\kappa = 1$. This last term can describes a form of redistribution of wealth across households. Each household has one unit of human-capital-adjusted time endowment to allocate between work, $l_t(\kappa)/h_t(\kappa)$, and human capital updating, $u_t(\kappa)$, at each time $t > 0$.[3] Hence, the human-capital-adjusted time constraint of a household κ is

$$1 = \frac{l_t(\kappa)}{h_t(\kappa)} + u_t(\kappa). \tag{3}$$

To preserve analytical tractability in the dynamic environment, we assume that, for each household κ and for all $t > 0$, the effort

[2] The framework can be readily extended to allow for population growth, without altering the main qualitative results (see Appendix G).

[3] As will be outlined in the following sections of the manuscript, the productivity of an intermediate good relies on its quality and the average quality of all intermediate goods capturing technology spillovers. Labor is considered the only R&D input that enables the productivity of each intermediate good to increase over time. Furthermore, the number of intermediate goods can grow with new entrants that produce at the average productivity. Consequently, it is assumed that individuals must allocate a portion of their own time to stay updated with the increasing know-how incorporated in the average productivity of all intermediate goods for the effective use of that productivity.

allocated to human capital investment is given by $u_t(\kappa) = uh_t$, where $u \in (0,1)$ captures the effectiveness of individual effort in updating skills relative to the frontier of aggregate knowledge. As will become clear below, aggregate human capital is proportional to the economy-wide stock of technical knowledge and productivity, denoted by Z_t. In this setting, households must continuously update their skills to keep pace with aggregate technological progress. This assumption admits a natural economic interpretation. Consider the human-capital-adjusted labor time constraint in equation (3), which can be written as

$$1 = \frac{l_t(\kappa)}{h_t(\kappa)} + \frac{uh_t}{h_t(\kappa)} = \frac{l_t(\kappa)}{h_t(\kappa)} + \frac{u}{s_{h,t}(\kappa)}, \tag{4}$$

where $s_{h,t}(\kappa) = h_t(\kappa)/h_t$ denotes household κ's share of aggregate human capital at time t. This expression highlights that households with a lower human capital share require more effort to remain technologically up to date. Consequently, from equation (3), the human-capital embodied labor time of each household κ is

$$l_t(\kappa) = h_t(\kappa) - uh_t. \tag{5}$$

The law of motion for human capital of a household κ is

$$\dot{h}_t(\kappa) = \xi(\kappa)\, u_t(\kappa) - \delta(\kappa)\, h_t, \tag{6}$$

where $\xi(\kappa) > 0$ is a productivity parameter for human capital updating, $\delta(\kappa) \geq 0$ is the depreciation rate of human capital of a household κ, and h_t is the aggregate (and average) human capital of households.

Each consumer maximizes eq. (1) choosing the optimal plan of $c_t(\kappa)$, $l_t(\kappa)$, subject to the law of motion of the asset equation and human capital as in equations (2) and (6) respectively, and to the time constraint as in eq. (5). From standard dynamic optimization, and using the current value Hamiltonian, the Euler equation is

$$\frac{\dot{c}_t(\kappa)}{c_t(\kappa)} = r_t - \rho, \tag{7}$$

where the interest rate is (see Appendix A)

$$r_t = \frac{\dot{w}_t}{w_t} + \xi(\kappa). \tag{8}$$

3.2 Final good

Following Chu et al. (2021), the final good Y_t is produced by competitive firms using the following production function:

$$Y_t = \int_0^{N_t} X_t^{\theta} \left[Z_t^{\alpha}(i) \, Z_t^{1-\alpha} \frac{E_t}{N_t} \right]^{1-\theta} di, \tag{9}$$

where $\{\theta, \alpha\} \in (0,1)$. $X_t(i)$ denotes the quantity of non-durable intermediate good $i \in [0, N_t]$, E_t denotes the quantity of energy services and N_t is the mass of available intermediate goods at time t, where homogeneous energy intensity of each intermediate good is assumed. The productivity of intermediate good $X_t(i)$ depends on its own quality $Z_t(i)$ and also on the average quality $Z_t = (1/N_t) \int_0^{N_t} Z_t(i) \, di$ of all intermediate goods capturing technology spillovers. The private return to quality is determined by α and the degree of technology spillovers is determined by $(1-\alpha)$.

The production function (9) also indicates that a higher productivity of an intermediate good i $(Z_t(i))$ and of average productivity (Z_t) reduce the energy services intensity (E_t/N_t), respectively with weights α and $(1-\alpha)$, needed to produce the same amount of the final good Y_t.

Profit maximization yields the following conditional demand functions for E_t and $X_t(i)$:

$$E_t = (1-\theta) \frac{Y_t}{p_t(e)}, \tag{10}$$

and

$$X_t(i) = \left(\frac{\theta}{p_t(i)} \right)^{\frac{1}{1-\theta}} Z_t^{\alpha}(i) \, Z_t^{1-\alpha} \frac{E_t}{N_t}, \tag{11}$$

where $p_t(e)$ and $p_t(i)$ are the price of E_t and $X_t(i)$ respectively. Competitive producers pay $(1-\theta) Y_t = p_t(e) E_t$ of final good for energy services and $\theta Y_t = \int_0^{N_t} p_t(i) X_t(i) \, di$ for intermediate goods. We adopt the small open economy assumption, and we assume that energy is in infinite net supply for this economy. In this way, the price of energy services $p_t(e)$ is taken as given and exogenously fixed.[4]

[4] Empirical studies have shown that mineral prices, including oil, coal and natural gas,

3.3 Intermediate goods and in-house R&D

The monopolistic firm in industry i produces the differentiated intermediate good with a linear technology that requires $X_t(i)$ units of final good to produce $X_t(i)$ units of intermediate good $i \in [0, N_t]$. Similarly to Chu et al. (2021), the firm in each industry i incurs σZ_t units of final good as a fixed operating cost, with $\sigma > 0$. To improve the quality of its product, the firm also employes flow labor time $L_t(i)$ to R&D. Therefore, the productivity of the industry i evolves according to the dynamic law[5]

$$\dot{Z}_t(i) = L_t(i). \tag{12}$$

In industry i, the monopolistic firm's (before-R&D) profit flow at time t is

$$\Pi_t(i) = [p_t(i) - 1] X_t(i) - \sigma Z_t. \tag{13}$$

The value of the monopolistic firm in industry i at time t is

$$V_t(i) = \int_t^{\infty} \exp\left(-\int r_s ds\right) [\Pi_s(i) - w_s L_s(i)] \, ds, \tag{14}$$

The monopolistic firm in industry i maximizes (14) subject to (11), (12) and (13). The current-value Hamiltonian for this optimization problem is

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have either been roughly trendless over time or have been stationary around deterministic trends with infrequent structural breaks (Lin and Wagner, 2007). The model set-up can be extended in a simple way to include an upstream sector producing energy from either fossil fuels or renewable sources. Yet, this extension does not add new insights to the paper. When considering the energy sector, the analysis described in the next sections can be replicated and the results and policy implications hold.

[5] Introducing nonlinearity in the R&D technology does not affect the qualitative properties of the model. We assume homogeneous research productivity across firms and normalize it to one. Minniti et al. (2013) and Marsiglio and Tolotti (2018) analyze the implications of heterogeneous research productivity within Schumpeterian growth frameworks; abstracting from such heterogeneity allows us to focus on the interaction between entry, firm scale, and aggregate innovation. With a constant population, entry of new intermediate firms engaged in R&D dilutes the fixed labor force across an expanding number of firms. As a result, the size of each firm's research laboratory declines as entry increases. To capture the idea that R&D requires a minimal organizational scale, we introduce an indivisibility in research activity: each intermediate firm must operate a laboratory of minimum size $L_{\min,t}(i) > 0$ in order to conduct effective R&D. Formally, innovation at the firm level is given by $\dot{Z}_t(i) = \begin{cases} L_t(i), & \text{if } L_t(i) \geq L_{\min,t}(i) \\ 0 & \text{if } L_t(i) < L_{\min,t}(i). \end{cases}$ We assume that population size is sufficiently large so that equilibrium laboratory size always exceeds $L_{\min,t}(i) > 0$. This assumption is not restrictive: the main results extend to an economy with population growth, in which the minimum-scale constraint never binds along the equilibrium path.

$$H_t(i) = \Pi_t(i) - w_t L_t(i) + \eta_t \dot{Z}_t(i), \tag{15}$$

where η_t is the co-state variable on (12). We solve this optimization problem in Appendix B and derive the unconstrained profit-maximizing markup ratio given by $(1/\theta)$. To analyze the effects of patent breadth, we introduce a policy parameter $\mu > 1$, which determines the unit cost for imitative firms to produce $X_t(i)$ with the same quality $Z_t(i)$ as the monopolistic firm in industry i.[6] In general, the parameter μ captures the market power of monopolistic firms. Here we consider the case, in which a larger patent breadth μ increases the production cost of imitative firms and allows the monopolistic producer of $X_t(i)$, who owns the patent, to charge a higher markup without losing her market share to potential imitators.[7]

Therefore, the equilibrium price becomes

$$p_t(i) = \min\{\mu, (1/\theta)\} \tag{16}$$

We follow previous studies to consider a symmetric equilibrium in which $Z_t(i) = Z_t$ and $L_t(i) = L_t$ for $i \in [0, N_t]$. In this case, the size of intermediate-good firms is also identical across all industries, such that $X_t(i) = X_t$.[8] From (11) and $p_t(i) = \mu$, the quality-adjusted firm size is

$$\frac{X_t}{Z_t} = \left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}} \frac{E_t}{N_t}. \tag{17}$$

We define the following transformed variable:

$$x_t \equiv \frac{X_t}{Z_t}(\mu)^{\frac{1}{1-\theta}} = (\theta)^{\frac{1}{1-\theta}} \frac{E_t}{N_t}, \tag{18}$$

where x_t is a state variable that is determined by the quality-adjusted firm size (X_t/Z_t), which in turn depends on the average energy use of all intermediate goods (E_t/N_t).

[6] Here we assume a diffusion of knowledge from the monopolistic firm to imitators.

[7] Intuitively, the presence of monopolistic profits attracts potential imitators. However, stronger patent protection increases the production cost of imitative products and allows monopolistic firms to charge a higher markup without losing market share to these potential imitators; see also Li (2001), Goh and Olivier (2002), Chu (2011) and Iwaisako and Futagami (2013) for a similar formulation.

[8] Symmetry also implies $\Pi_t(i) = \Pi_t$ and $V_t(i) = V_t$.

3.4 Entrants

Following previous studies, we assume that entrants have access to aggregate technology Z_t to ensure symmetric equilibrium at any time t. A new firm pays βX_t units of final good to set up its operation and enter the market with a new product (which will be protected by a patent). $\beta > 0$ is a cost parameter, and the cost function βX_t captures the case in which the setup cost is increasing in the initial output volume of the firm.

In a symmetric equilibrium, the asset-pricing equation determines the rate of return on assets as

$$r_t = \frac{\Pi_t - w_t L_t}{V_t} + \frac{\dot{V}_t}{V_t}. \tag{19}$$

Intuitively, the asset-pricing equation equates the interest rate to the rate of return from V_t, which is given by the monopolistic profit net of the R&D cost $w_t L_t$ plus the capital gain $\dot{V}_t$. The free-entry condition is given by[9]

$$V_t = \beta X_t. \tag{20}$$

Because the size of intermediate-good firms is identical across all industries, i.e., $X_t(i) = X_t$, from (11), $p_t(i) = \mu$, and condition (18), the free entry condition (11) becomes

$$V_t = \beta \left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t. \tag{21}$$

Substituting (12), (13), (18), (21), and $p_t(i) = \mu$ into (19) yields the return on entry as

$$\begin{aligned} r_t = &\frac{(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t - \sigma Z_t}{\beta\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t} + \\ &- \frac{w_t L_t}{\beta\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t} + z_t + \frac{\dot{x}_t}{x_t}, \end{aligned} \tag{22}$$

from which the quality-adjusted firm size is obtained (see Appendix C):

$$x_t = \frac{\sigma}{\beta} (\mu)^{\frac{1}{1-\theta}} \left[\frac{(\mu-1)}{\beta} + z_t \left(1 - \frac{\alpha}{\xi}\frac{\mu-1}{\beta}\right) - \xi\right]^{-1}. \tag{23}$$

[9] Following Chu et al. (2021), we treat entry and exit symmetrically (i.e., the scrap value of exiting an industry is also βX_t); therefore, $V_t = \beta X_t$ always holds. If $V_t > \beta X_t$ ($V_t < \beta X_t$), then there would be an infinite number of entries (exits).

4 General equilibrium

The equilibrium is a time path of allocations

$\{c_t(\kappa), a_t(\kappa), h_t(\kappa), l_t(\kappa), N_t, Y_t, X_t, E_t, L_t\}$,

and a time path of prices $\{p_t(i), w_t, p_t(e), r_t, V_t(i)\}$. Also, at each instance of time t, the following holds:

- households maximize utility taking $\{w_t, r_t\}$ as given;
- competitive final-goods firms produce Y_t to maximize profit taking $\{p_t(i), p_t(e)\}$ as given;
- monopolistic firms produce $X_t(i)$ and choose $\{L_t(i), p_t(i)\}$ to maximize profit taking $\{w_t, r_t\}$ as given;
- entrants make entry decisions taking V_t as given;
- the value of all existing monopolistic firms adds up to the value of the households' assets such that $N_t V_t = \int_0^1 a_t(\kappa)\, dF(\kappa) \equiv a_t$;
- the market-clearing condition for human capital-embodied labor supply holds such that $\int_0^1 l_t(\kappa)\, d\kappa = \int_0^1 h_t(\kappa)\, dF(\kappa) - u h_t = (1-u) h_t = L$;
- the market-clearing condition for final goods holds such that

$$Y_t = c_t + N_t(X_t + \sigma Z_t) + \dot{N}_t \beta X_t. \tag{24}$$

5 Dynamics

In this section, we analyze the dynamics of the model. Section 4.1 presents the dynamics of the aggregate economy. Section 4.2 summarizes the dynamics of human capital and of wealth distribution. Section 4.3 derives the time evolution of the income distribution and of the Gini index of income.

5.1 Dynamics of the aggregate economy

We now analyze the dynamics of the economy. To this aim, we have to be equipped with the growth rate of the aggregate human capital of the economy. Using the law of motion of human capital of a household κ (6), the aggregate human capital accumulation of the economy is

$$\begin{aligned}\dot{h}_t = \textstyle\int_0^1 \dot{h}_t(\kappa)\, dF(\kappa) = \\ = h_t \textstyle\int_0^1 (\xi(\kappa)\, u - \delta(\kappa))\, dF(\kappa) = h_t(\xi u - \delta),\end{aligned} \tag{25}$$

where $\xi = \int_0^1 \xi(\kappa)\, dF(\kappa)$, $\delta = \int_0^1 \delta(\kappa)\, dF(\kappa)$, and $F(\bullet)$ is a continuous cumulative distribution function of human capital of households exogenously given, with usual properties, i.e., $F(0) = 0$, $F(1) = 1$, and $F'(\kappa) > 0$. To facilitate notation and exposition, and with no loss of generality in the analysis, we assume that $F(\kappa)$ is a uniform distribution on $\kappa \in [0, 1]$. Therefore, the rate of human capital becomes:

$$g_h \equiv \frac{\dot{h}_t}{h_t} = \xi u - \delta, \tag{26}$$

which is stationary. Condition (12) implies the growth rate of aggregate quality, $z_t = g_h = (\xi u - \delta)$, is stationary. Equation (23) implies that the quality-adjusted firm size x_t is stationary and equal to

$$x_t = \frac{\sigma}{\beta}(\mu)^{\frac{1}{1-\theta}} \left[\frac{(\mu - 1)}{\beta} + (\xi u - \delta)\left(1 - \frac{\alpha}{\xi}\frac{\mu - 1}{\beta}\right) - \xi\right]^{-1}. \tag{27}$$

The wage flow is also stationary and equal to (see Appendix B)

$$w_t = \frac{\alpha}{\xi}\left[(\mu - 1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} x_t\right]. \tag{28}$$

Moreover, because condition $(\dot{w}_t / w_t) = r_t - \xi = 0$ holds, the interest rate is also stationary and equal to $r_t = \xi$. Substituting (13), (18) into (14), the monopolistic firm stock market value becomes:

$$\begin{aligned}V_t = \textstyle\int_t^\infty e^{-r_s \bullet s}\left(\Pi_s - w_s L_s(i)\right) ds = \\ = \textstyle\int_t^\infty e^{-\rho s}\left[\left((\mu - 1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} x_s - \sigma\right) Z_s - w_s L_s(i)\right] ds,\end{aligned} \tag{29}$$

where $L_s(i)$ indicates the the human-capital embodied labor employed by the monopolistic firm i. The quality-adjusted firm size x_t and the wage flow per unit of human capital w_t are constants over time, while the aggregate technology Z_t and the aggregate (and average) human-capital embodied labor L_t grow at the common rate $z_t = g_h = (\xi u - \delta)$. Therefore, equation (29) can be written as:

$$V_t = [\exp(\xi u - \delta - \xi) \bullet t] \times \\ \times \frac{\left[\left((\mu-1)\left(\frac{1}{\mu}\right)^{(1/(1-\theta))} x_t - \sigma\right) Z_0 - w_t L_0(i)\right]}{\xi - (\xi u - \delta)}, \tag{30}$$

where inequality $\xi - (\xi u - \delta) = \xi(1-u) + \delta > 0$ holds because $u \in (0,1)$. The value of the households' assets is $a_t = V_t N_t$, so that

$$\frac{\dot{a}_t}{a_t} = \frac{\dot{V}_t}{V_t} + \frac{\dot{N}_t}{N_t} = z_t + \frac{\dot{N}_t}{N_t}, \tag{31}$$

where we have use equation (21) to get $\dot{V}_t/V_t = z_t$, which is consistent with condition (29).

Using (11) and assuming symmetry, the aggregate final output is

$$Y_t = \left(\frac{\theta}{\mu}\right)^{\frac{\theta}{1-\theta}} Z_t E_t. \tag{32}$$

Because the quality-adjusted firm size (x_t) is constant, we have $\dot{E}_t/E_t = \dot{N}_t/N_t$, and the growth rate of final output is: $g_Y \equiv \dot{Y}_t/Y_t = z_t + \dot{N}_t/N_t$. Consequently, condition

$$\frac{\dot{a}_t}{a_t} = \frac{\dot{Y}_t}{Y_t} = z_t + \frac{\dot{N}_t}{N_t}, \tag{33}$$

holds. New firms enter in the market according to the law of motion:

$$\dot{N}_t = \frac{u L_t}{N_t^{\phi}}, \tag{34}$$

where ϕ is a measures of the negative (congestion) or positive (spillover) effect in creating new intermediate goods generated by the already existing intermediate goods N_t for a given stock of updated human capital of individuals, uL_t. The law of motion of the number of monopolistic firms (34) generates the growth rate of intermediate goods

$$\frac{\dot{N}_t}{N_t} = \frac{g_h}{1+\phi}. \tag{35}$$

Condition (35) determines the growth rate of the financial asset per capita and of final output

$$\frac{\dot{a}_t}{a_t} = \frac{\dot{Y}_t}{Y_t} = z_t + \frac{g_h}{1+\phi} = \left(\frac{2+\phi}{1+\phi}\right)(\xi u - \delta), \tag{36}$$

where we have used $z_t = g_h = \xi u - \delta$.[10]

5.2 Dynamics of human capital and wealth distribution

In this section, starting with the dynamics of human capital as in equation (26), we show the dynamics of the share of human capital and of wealth distribution.

Let $s_{h,t}(\kappa) \equiv h_t(\kappa)/h_t$ denote the share of human capital of a household κ. Then, the growth rate of $s_{h,t}(\kappa)$ is given by

$$\dot{s}_{h,t}(\kappa) = \left(\frac{\dot{h}_t(\kappa)}{h_t(\kappa)} - \frac{\dot{h}_t}{h_t}\right) s_{h,t}(\kappa) = \frac{\dot{h}_t(\kappa)}{h_t} - g_h s_{h,t}(\kappa). \tag{37}$$

Using again the law of motion of human capital of a household κ (6), we have $\dot{h}_t(\kappa)/h_t = \xi(\kappa) u - \delta(\kappa)$, and $g_h = \xi u - \delta$ Therefore, the share of human capital of a household κ is

$$\dot{s}_{h,t}(\kappa) = \xi(\kappa) u - \delta(\kappa) - (\xi u - \delta) s_{h,t}(\kappa). \tag{38}$$

The differential equation (38) admits a unique solution that describes the time trajectory of the share of human capital of a household κ as

$$\begin{gathered} s_{h,t}(\kappa) = \tfrac{\xi(\kappa)u-\delta(\kappa)}{\xi u-\delta} + \\ -e^{-(\xi u-\delta)t}\left(\tfrac{\xi(\kappa)u-\delta(\kappa)}{\xi u-\delta} - s_{h,0}(\kappa)\right). \end{gathered} \tag{39}$$

The time path of household κ's share of human capital increases over time if its initial share is sufficiently large, that is, if $s_{h,0}(\kappa) >$

[10] If the minimum laboratory size were to become binding, firm entry $\dot{N}_t$ would continue only until aggregate R&D labor L_t is just sufficient to operate all active firms at the homogeneous minimum efficient scale $L_{\min,t}(i) > 0$. Entry would therefore occur whenever $L_t > N_t L_{\min,t}$ and would cease at the threshold $N_t = L_t/L_{\min,t}$, implying a stationary mass of firms $N_t = L_t/L_{\min,t}$. In this case, per capita financial wealth and output would grow at the common rate $\dot{a}_t/a_t = \dot{Y}_t/Y_t = z_t = \xi u - \delta$. The qualitative properties of wealth distribution dynamics and the evolution of the Gini coefficient remain unchanged relative to the benchmark specification with endogenous entry.

$[\xi(\kappa) u - \delta(\kappa)] / (\xi u - \delta)$. Otherwise, the household's share of human capital declines over time. It is worth noting that the transmission of advantage across generations has far-reaching implications as to how we view current levels of inequality and the degree of equality of opportunity in a society. Even if recent research on intergenerational mobility suggests that the traditional child-parent regression model underestimates long-term persistence, to date there is still no consensus on how the results should be interpreted, implying very different views on what level of social mobility, ranging from fairly high levels of mobility to almost perfect persistence, a society is facing.[11] Recently, Adermon, Lindahl and Palme (2021) provide a framework for estimating long-run intergenerational persistence using direct measures based on observed extended family relations (the dynasty) of the entire Swedish population. Using various human capital measures, the authors show that traditional parent-child estimates underestimate long-run intergenerational persistence by at least one-third. The results of Adermon, Lindahl and Palme (2021) are found to be robust to different extensions to the main analysis.

Aggregating (2) across all households yields the following aggregate asset accumulation equation:

$$\begin{gathered} \dot{a}_t = r_t a_t + w_t (1-\tau) L_t + \tau w_t L_t - c_t = \\ r_t a_t + w_t L_t - c_t \end{gathered} \tag{40}$$

where $\int_0^1 \psi(\kappa) d\kappa = 1$, $L_t = \int_0^1 l_t(\kappa) d\kappa$. Let $s_{a,t}(\kappa) \equiv a_t(\kappa)/a_t$ denote the share of wealth owned by household κ. Then, the growth rate of $s_{a,t}(\kappa)$ is given by

$$\begin{gathered} \frac{\dot{s}_{a,t}(\kappa)}{s_{a,t}(\kappa)} = \left(\frac{\dot{a}_t(\kappa)}{a_t(\kappa)} - \frac{\dot{a}_t}{a_t}\right) = \\ = \frac{r_t a_t(\kappa) + (1-\tau) w_t l_t(\kappa) + \psi(\kappa) \tau w_t L_t - c_t(\kappa)}{a_t(\kappa)} - \frac{r_t a_t + w_t L_t - c_t}{a_t}, \end{gathered} \tag{41}$$

that can be rewritten as:

[11] See, e.g., Chan and Boliver (2013), Lindahl et al. (2015), Braun and Stuhler (2018), and Long and Ferrie (2018), who all use data in which families have been linked through multiple generations, and Clark (2014), Clark and Cummins (2015), and Barone and Mocetti (2016), for example, who use data in which generations have been linked through surnames. Some are skeptical of the new findings, arguing that the influence of ancestors may be spurious and that results based on surnames estimate a different parameter from the one obtained from traditional child-parent regressions (Solon 2018).

$$\begin{aligned}\dot{s}_{a,t}(\kappa) = \tfrac{c_t - w_t L_t}{a_t} s_{a,t}(\kappa) + \tfrac{(1-\tau) w_t l_t(\kappa) + \psi(\kappa)\tau w_t L_t - s_{c,t}(\kappa) c_t}{a_t} = \\ = \tfrac{c_t}{a_t}\left(s_{a,t}(\kappa) - s_{c,t}(\kappa)\right) + \\ -\tfrac{w_t}{a_t}\left[L_t\left(s_{a,t}(\kappa) - \psi(\kappa)\tau\right) - (1-\tau) l_t(\kappa)\right]\end{aligned} \quad (42)$$

where we have defined the share of consumption owned by household κ to be $s_{c,t}(\kappa) \equiv c_t(\kappa)/c_t$. From the Euler equation (7), and given the stationarity of the interest rate, $r_t = \xi$, the consumption of each household κ changes at a constant and common rate, i.e., $\dot{c}_t(\kappa)/c_t(\kappa) = r_t - \xi$. This implies the stationarity of the consumption share $s_{c,t}(\kappa)$, i.e., $\dot{s}_{c,t}(\kappa) = (\dot{c}_t(\kappa)/c_t(\kappa) - \dot{c}_t/c_t)\, s_{c,t}(\kappa) = 0$.

Considering that $L_t = \int_0^1 l_t(\kappa)\, d\kappa = \int_0^1 [h_t(\kappa) - uh_t]\, d\kappa = h_t(1-u)$, and $l_t(\kappa) = h_t(\kappa) - uh_t$, condition (42) can be rewritten as

$$\begin{aligned}\dot{s}_{a,t}(\kappa) = \tfrac{c_t}{a_t}\left(s_{a,t}(\kappa) - s_{c,t}(\kappa)\right) + \\ -\tfrac{w_t h_t}{a_t}\left[(1-u)\left(s_{a,t}(\kappa) - \psi(\kappa)\tau\right) - (1-\tau)\left(s_{h,t}(\kappa) - u\right)\right],\end{aligned} \quad (43)$$

where $a_t = a_0 e^{\left[z_t + \frac{g_h}{1+\phi}\right]*t}$, $h_t = h_0 e^{g_h t}$, $g_h = z_t = (\xi u - \delta)$, $c_t = c_0$, $L_t = L_0 e^{g_h t}$, w_t is constant as in (28).

The law of motion for an individual household's wealth share admits a clear economic interpretation. The first term on the right-hand side of equation (43) reflects the role of relative consumption behavior. When a household's wealth share exceeds its consumption share - that is, when $s_{a,t}(\kappa) - s_{c,t}(\kappa) > 0$ - the household accumulates assets at a rate above the aggregate average, thereby increasing its share of total wealth. The strength of this mechanism is weighted by the ratio of aggregate consumption to aggregate asset holdings, c_t/a_t. Indeed, a higher value of c_t/a_t amplifies the relative advantage in asset accumulation enjoyed by households whose wealth share is large relative to their consumption share. The second term reflects the contribution of labor income relative to asset values. Specifically, labor earnings, measured by $(w_t h_t)$, affect wealth accumulation in proportion to their importance relative to aggregate wealth, $(w_t h_t)/a_t$. When a household's share of human capital net of skill-updating time is sufficiently large, $s_{h,t}(\kappa) - u > 0$, and exceeds its current wealth share, labor income contributes positively to the growth of the household's wealth share. This term highlights how heterogeneity in human capital shapes the dynamics of wealth inequality by differentially linking labor income to asset accumulation across households. It is worth noting that a higher aggregate asset value, $a_t = N_t V_t$, plays an important role in shaping wealth inequality, as changes in aggregate asset valuations alter the relative contribution of labor income and asset income to wealth accumulation across households.

Since the functions on the right-hand side of the differential equation (43) are continuous, standard results guarantee the existence and uniqueness of a solution to (43) for any given initial condition (see, e.g., Boyce et al., 2021). However, because the explicit solution depends on a large set of parameter values, deriving analytical expressions that yield clear economic insights is infeasible. We therefore analyze the time path of the system numerically in the remainder of the manuscript.

5.3 Time evolution of the income distribution

In this section, we show that the income distribution is endogenous and nonstationary. Income received by household κ is given by

$$I_t(\kappa) = r_t a_t(\kappa) + w_t(1-\tau) l_t(\kappa) + \psi(\kappa)\tau w_t L_t. \tag{44}$$

Aggregating (44) yields the aggregate level of income as

$$I_t = r_t a_t + w_t L_t. \tag{45}$$

Let $s_{I,t}(\kappa) \equiv I_t(\kappa)/I_t$ denote the share of income received by household κ. Then,we have

$$\begin{aligned} s_{I,t}(\kappa) &= \tfrac{r_t a_t(\kappa) + (1-\tau) w_t l_t(\kappa) + \psi(\kappa)\tau w_t L_t}{r_t a_t + w_t L_t} = \\ &= \tfrac{r_t a_t}{r_t a_t + w_t h_t (1-u)} s_{a,t}(\kappa) + \tfrac{w_t h_t \left[(1-\tau)\left(s_{h,t}(\kappa) - u\right) + \psi(\kappa)\tau(1-u)\right]}{r_t a_t + w_t h_t(1-u)}, \end{aligned} \tag{46}$$

where $\int_0^1 \psi(\kappa)\, d\kappa = 1$. Equation (46) determines the evolution of the share of income received by household κ and allows us to derive any moment of the income distribution.

The equation governing the evolution of a household's income share also admits a clear economic interpretation. The first term on the right-hand side of equation (46) reflects the contribution of asset ownership: a higher share of aggregate wealth directly translates into a larger share of total income through capital returns.

The second term captures the role of labor income and redistribution. Labor income contributes positively to the income share when the household's share of human capital net of skill-updating time is sufficiently large, $s_{h,t}(\kappa) - u > 0$, thereby increasing earnings relative to the aggregate. In addition, this term incorporates the redistributive component of the Social Security system, represented by the household's share of aggregate Social Security transfers, $\psi(\kappa)$, which reallocates income across households independently of asset ownership. Together, these channels clarify how income shares are jointly shaped by wealth concentration, human capital heterogeneity, and public redistribution.

Following Chu et al. (2021), the Gini coefficient of income at any time t is given by

$$\begin{gathered}\sigma_{I,t} = \tfrac{r_t a_t}{r_t a_t + w_t h_t (1-u)} \sigma_{a,t} + \\ + \tfrac{w_t h_t}{r_t a_t + w_t h_t (1-u)} \left[(1-u)(1-\tau)\left(\tfrac{1}{3} - \tfrac{1}{2}u\right) + \tfrac{1}{3}\tau(1-u) \right],\end{gathered} \tag{47}$$

where $\sigma_{a,t}$ represents the Gini coefficient of wealth (see Appendix D).

5.4 Quantitative Analysis

As discussed in the previous sections, the differential equation governing the evolution of household κ's share of aggregate wealth (43) admits a closed-form analytical solution. However, because the explicit solution depends on a large set of parameter values, the dynamics of the model through numerical simulations. To this end, the model is calibrated to aggregate U.S. data to conduct a quantitative analysis because all needed data are available for this country (see Appendix E for details).

The parameter set of the model is $\{\alpha, \xi, \mu, \theta, u, \delta, \phi, \rho, \beta, \sigma\}$. Following Iacopetta et al. (2019), we set the degree of technology spillovers, $1 - \alpha$, equal to 0.833. The interest rate, denoted by r_t, is set to the conventional value of 0.0331, which implies a real interest rate of 3.31 percent. This figure is consistent with the long-term average real interest rate for the period 2000-2021, as reported by the OECD Statistics database (annual percentage). It is important to emphasize that alternative specifications of r_t do not qualitatively alter the calibration results. Since the model optimal conditions of a consumption imply $r_t = \xi$, the value of parameter ξ is set at 0.0331. The markup parameter μ is fixed at 1.38, corresponding to the average markup estimated by Hall (2018) for the U.S. economy over 1988–2015. The parameter θ is calibrated to satisfy the condition $\mu < 1/\theta$. To ensure robustness, we follow Hall (2018) and consider the highest estimated markup, $\mu = 1.85$, observed in the Accommodation and Food Services sector. Imposing $\theta < 1/\mu$ yields $\theta < (1/1.85) \simeq 0.54$. We therefore set $\theta = 0.5$, which satisfies this constraint and preserves internal consistency within the model. Following Tenopir et al. (2012), the updating time parameter u is set equal to 0.21.

We calibrate the depreciation rate of human capital δ using the model-implied growth rate of human capital, $g_h = (\xi u - \delta)$, targeting its empirical counterpart. Following Mulligan and Sala-i-Martin (2000), we set the average annual growth rate of human capital in the U.S. economy to 0.4 percent. The productivity parameter governing human capital accumulation, ξ, equals the interest rate $r_t = \xi = 0.0331$. Then the value of the the depreciation rate of human capital $\delta = 0.00295$. The congestion parameter in the production of new intermediate varieties, ϕ, is calibrated to ensure that the model-implied growth rate of GDP per capita, $g = (\xi u - \delta)(2 + \phi) / (1 + \phi)$ matches its empirical counterpart. According to the World Development Indicators (WDI, 2023), the average growth rate of U.S. GDP per capita over the period 1976-2022 is 1.8 percent, which the model reproduces under the chosen parameterization. The implied value of the congestion parameter is $\phi = -0.714$. The subjective discount rate ρ is calibrated using the Euler equation for consumption $\dot{c}_t(\kappa) / c_t(\kappa) = r_t - \xi$. Since the model specification implies that the growth rate of the consumption per capita is equal to the growth rate of the GDP per capita $g = 0.018$, the value of the subjective discount rate is set at $\rho = \xi - g = 0.0331 - 0.018 = 0.0151$.

Conditional on the parameter values discussed above, the remaining unknowns in the equilibrium conditions are β and σ. These parameters are calibrated using the firm-size condition (27) and the wage equation (28). Specifically, β and σ are chosen so that the simulated wage flow equals its normalized initial level, $w_0 = 1$, consistent with a stationary wage in equilibrium. The resulting calibrated values are $\beta = 0.01$ and $\sigma = 0.194$. Sensitivity analyses indicate that alternative parameterization do not qualitatively affect either the calibration results or the dynamic properties of wealth and income distributions.

To characterize the time path of the average wealth share in the economy, we partition the wealth distribution into four groups: households in the bottom 25 percent, those between the 26th and 50th percentiles, those between the 51st and 90th percentiles, and households in the top 10 percent. The data are drawn from the U.S. Congressional Budget Office (CBO, Trends in the Distribution of Family Wealth, 2024) and cover the period 1989-2019.

Aggregate wealth, a_t, is computed as the sum of average wealth holdings across these four percentile groups, weighted by their respective population shares. The model-implied time path of the average wealth share follows equation (42), where the initial level of average wealth, $a_0\,(avg)$, is defined as the percentile-weighted average of household wealth across groups. Using the CBO data, this initial value is set to $a_0\,(avg) = 9.5$, with the initial period of the analysis corresponding to 1989. Consistently, the initial average wealth share, $s_{a,0}\,(avg)$, is constructed as the weighted sum of wealth shares held by each percentile group relative to total wealth. Using the CBO data, this initial value is set to $s_{a,0}\,(avg) = 0.146$.

Notably, the qualitative features of this trajectory exhibit a high degree of robustness across a wide range of parameter values governing the underlying differential equation (43).

Given the calibrated parameter values, we compare the time evolution of wealth shares generated by the numerical simulation with their empirical counterparts. Evidence reported by Horowitz et al. (2020) documents a substantial widening of the wealth gap between upper-, middle-, and lower-income households. Between 2001 and 2016, only upper-income households experienced sustained wealth accumulation, with median net worth increasing by 33%. In contrast, median net worth declined by 20% for middle-income households and by 45% for lower-income households. By 2016, upper-income households held 7.4 times the wealth of middle-income households and 75 times that of lower-income households, compared with ratios of 3.4 and 28, respectively, in 1983.[12] Consistent with

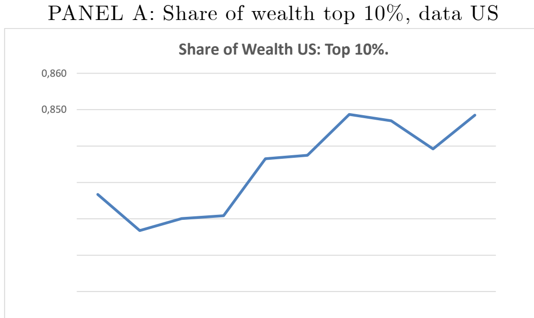
[12] These divergent trends reflect differences in portfolio composition. Middle-income house-

these patterns, the share of aggregate U.S. wealth held by upper-income households rose markedly - from 60% in 1983 to 79% in 2016 - while the share held by middle-income households declined from 32% to 17%.

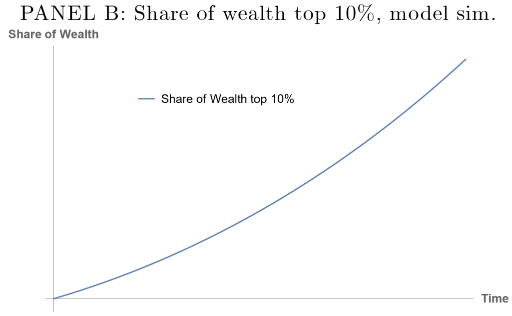
Figures 1 and 2 compare the model's quantitative implications with the data. Figure 1 reports the dynamic trajectory of the wealth share of the top 10 percent of the population, while Figure 2 focuses on the 50th percentile. In each figure, Panel A presents the empirical series for the United States over the period 1989–2016, drawn from the World Inequality Database (2025), and Panel B displays the corresponding paths generated by the calibrated model.

Figure 1

PANEL A: Share of wealth top 10%, data US



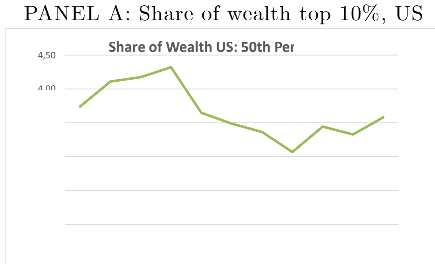
PANEL B: Share of wealth top 10%, model sim.



holds are more heavily exposed to housing wealth and were disproportionately affected by the collapse of the housing bubble in 2006. Upper-income households, whose portfolios are more concentrated in financial assets and business equity, benefited more from the subsequent recovery in equity markets.

Figure 2

PANEL A: Share of wealth top 10%, US



PANEL B: Share of wealth 50th P., US

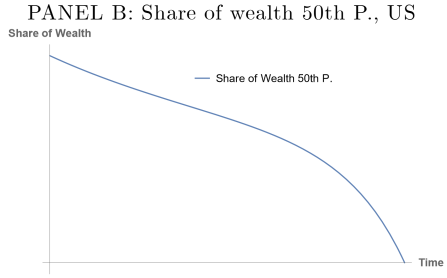
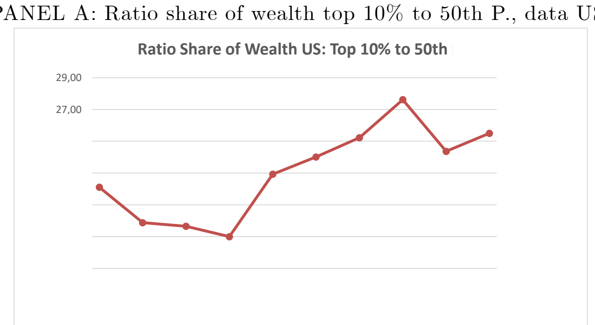


Figure 3 reports the time path of the ratio between the wealth share of the top 10 percent and that of the 50th percentile of the population. Panel A presents the corresponding empirical series for the United States over the period 1989–2016, drawn from the World Inequality Database (2025). Panel B displays the analogous trajectory generated by the calibrated model through numerical simulation.

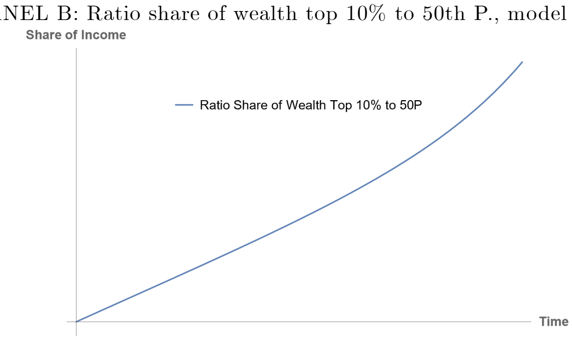
This comparison provides a direct assessment of the model's ability to replicate the relative concentration of wealth at the top of the distribution over time.

Figure 3

PANEL A: Ratio share of wealth top 10% to 50th P., data US.



PANEL B: Ratio share of wealth top 10% to 50th P., model sim..



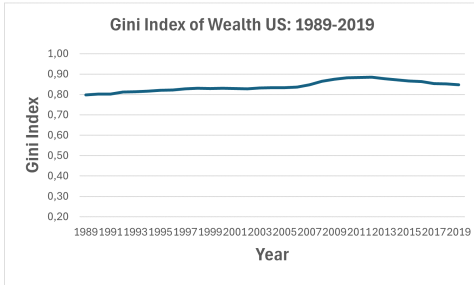
5.5 Wealth and Income Inequality

We compare the trajectories of wealth and income inequality generated by the model - measured by their respective Gini indices - with the corresponding empirical time paths for the U.S. economy. The model-implied time series for the income Gini index is constructed using equation (47), where $\sigma_{a,t}$ denotes the Gini index of wealth. The term $s_{a,t}(\kappa)$ represents the evolving average wealth share of household κ, which follows the dynamic law specified in equation (43).

Figure 4 presents the time path of the wealth Gini index. Panel A reports the empirical Gini series for the United States over the period 1989-2019, obtained from the World Inequality Database (2025). Panel B displays the corresponding dynamic trajectory generated by the model.

Figure 4

PANEL A: Gini Index of Wealth US 1989-2019



PANEL B: Gini Index of Wealth US 1989-2019

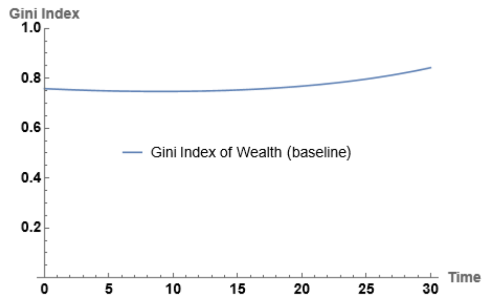
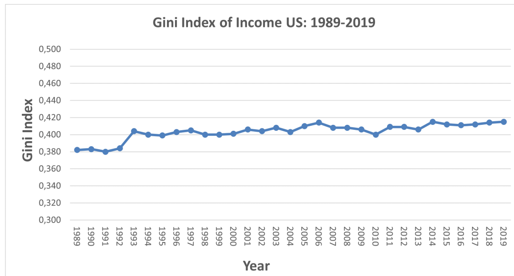


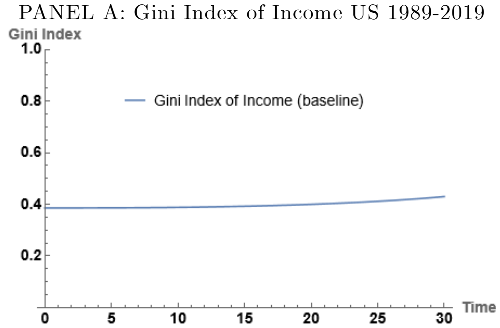
Figure 5 presents the time path of the income Gini index. Panel A reports the empirical Gini series for the United States over the period 1989-2019, obtained from the World Development Indicators (World Bank, 2025). Panel B displays the corresponding dynamic trajectory generated by the model.

FIGURE 5

PANEL A: Gini Index of Income US 1989-2019



PANEL A: Gini Index of Income US 1989-2019



6 Comparative Statics

This section analyzes the effects of patent breadth μ and of a change in the time devoted to updating, u, on wealth and income inequality. To this aim, both theoretical results and quantitative analysis are used.

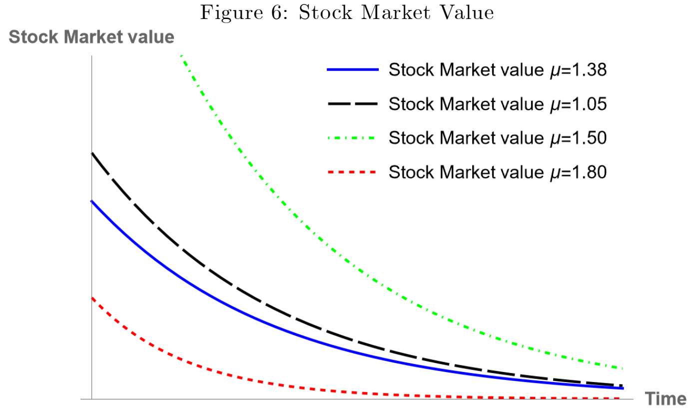
6.1 Strengthening patent protection

This section examines the effects of patent breadth on wealth and income inequality. The model predicts that an expansion in patent breadth, by raising the price of each intermediate variety, reduces productivity-adjusted firm size, x_t. At the same time, the reduction in incumbents' market shares lowers the entry cost of new firms, βX_t, thereby stimulating entry by producers of new patented varieties. This endogenous adjustment in market structure generates opposing effects on the demand for skilled labor. On the one hand, smaller firm-level market shares reduce the demand for skilled workers within each incumbent firm. On the other hand, entry of firms producing new intermediate varieties increases labor demand for R&D activities. Through these channels, patent breadth exerts offsetting effects on the wage flow per unit of human capital, w_t (see Appendix F).

These countervailing forces give rise to a non-monotonic relationship between patent breadth and the expected stock market value of patented products. An increase in patent breadth, which reduces the productivity-adjusted size of incumbent firms, x_t, tends to depress their market value, V_t, by compressing the market share associated with each variety. As profitability declines, firms reduce labor demand in R&D, exerting downward pressure on wages and research costs. This adjustment partially mitigates the adverse effect of smaller firm size and reduced market share on V_t. At the same time, stronger patent protection affects the entry margin. Entry by new intermediate producers expands the mass of firms engaged in R&D, increasing aggregate labor demand and wage payments. This general equilibrium effect offsets the decline in firm-level R&D costs, thereby shaping the overall response of firm value to changes in patent breadth.

Numerical simulations show that when patent protection is strengthened only mildly, the positive effect of lower R&D labor costs, $w_t L_t(i)$, dominates the negative effect of reduced market shares, leading to an increase in firms' expected stock market values. In this case, a modest increase in patent breadth slightly reduces firm size and market shares as well as the entry cost of new firms, resulting in limited entry and weaker upward pressure on R&D labor demand and wages. Consequently, R&D labor costs fall sufficiently to raise firm valuations. By contrast, when patent breadth is strengthened further, firm size and market shares decline substantially and entry costs fall sharply, triggering a stronger expansion in firm entry, and the negative effect of lower market shares dominates the positive effect of lower R&D labor cost. As a result, firms' expected stock market values decline (see Figure 6).

Figure 6: Stock Market Value



This non-monotonic effect of patent breadth on firms' asset valuation also has non-monotonic effect on wealth and income inequality. To facilitate reading, the dynamic equation governing the time evolution of households' share of wealth is here reported

$$\begin{gathered}\dot{s}_{a,t}\left(\kappa\right)=\tfrac{c_t}{a_t}\left(s_{a,t}\left(\kappa\right)-s_{c,t}\left(\kappa\right)\right)+\\ -\tfrac{w_t h_t}{a_t}\left[\left(1-u\right)\left(s_{a,t}\left(\kappa\right)-\psi\left(\kappa\right)\tau\right)-\left(1-\tau\right)\left(s_{h,t}\left(\kappa\right)-u\right)\right],\end{gathered} \tag{48}$$

The first term in equation (48) captures the role of the household's share wealth relative to its share of consumption weighted by the aggregate consumption to asset value on wealth accumulation. When a household's wealth share exceeds its consumption share - that is, when $s_{a,t}\left(\kappa\right)-s_{c,t}\left(\kappa\right)>0$ - the household accumulates assets at a faster pace. The strength of this effect is governed by the ratio of aggregate consumption to aggregate asset holdings, c_t/a_t. A higher value of c_t/a_t magnifies the relative advantage in asset accumulation enjoyed by households whose wealth share exceeds their consumption share, as it increases the responsiveness of wealth dynamics to saving differentials. Changes in aggregate asset values play a central role in this mechanism. An increase in a_t - for instance, following a moderate strengthening of patent protection - reduces the ratio c_t/a_t and therefore attenuates the relative advantage of the wealthiest households in accumulating assets. By contrast, a decline in a_t, as may occur under sufficiently strong patent breadth, raises c_t/a_t and amplifies the asset accumulation advantage of households at the top of the wealth distribution.

However, a central mechanism governing the time evolution of wealth inequality operates through the relative importance of labor income and asset income in households' wealth dynamics. When a household's share of human capital at time t is sufficiently high, $s_{h,t}(\kappa) > u$, labor income relative to aggregate asset value, $(w_t h_t)/a_t$, contributes positively to the household's wealth share. In this case, a higher human capital share implies a stronger positive effect of labor earnings, relative to asset holdings, on wealth accumulation. A moderate strengthening of patent protection, by increasing aggregate asset values a_t, attenuates this positive contribution: as asset values rise, the impact of labor income relative to assets on wealth accumulation diminishes, i.e., the term $(w_t h_t)/a_t$ is lower, and this gives an advantage to households with a relatively high share of wealth with respect to their own share of human capital. Since wealthier households typically hold a higher share of wealth relative to their share of human capital, this mechanism implies that a moderate increase in patent breadth increases wealth accumulation among richer households. By contrast, when a household's share of human capital at time t is low, $s_{h,t}(\kappa) < u$, the same term exerts a negative effect on the household's wealth share. Lower human capital shares imply a stronger negative contribution of labor earnings, relative to asset values, to wealth accumulation. In this case as well, a moderate strengthening of patent protection that raises aggregate asset values a_t mitigates the magnitude of this negative effect by reducing the importance of labor income relative to assets, i.e., the term $(w_t h_t)/a_t$ is lower, and this attenuates the decrease of the share of wealth of the poorer households. This interaction explains the slight positive effect of a modest strengthening of patent protection on wealth inequality. Figure 8, Panel A, illustrates this scenario for a moderate increase in patent protection, in which the average markup rising from its baseline value of $\mu = 1.38$ to $\mu = 1.50$. However, as time elapses, human capital accumulation allows the poorer households to increase their share of wealth, which is further spurred by the high asset value a_t.[13] This explains the moderate increase of wealth inequality over a longer time horizon with a modest

[13] The dynamic equation of the household's human capital accumulation, $\dot{h}_t(\kappa) = \xi(\kappa) u_t(\kappa) - \delta(\kappa) h_t$, implies that the growth rate of a household's human capital is higher the lower its own share of human capital, $\dot{h}_t(\kappa)/h_t(\kappa) = [\xi(\kappa) u - \delta(\kappa)]/s_{h,t}(\kappa)$.

strengthening of patent protection.

By contrast, when patent protection is tightened substantially - as shown in Figure 7, Panel A, where the average markup increases from $\mu = 1.38$ to $\mu = 1.80$ - the mechanisms described above imply that the resulting decline in aggregate asset values, a_t, decreases wealth inequality, at least initially. However, as time elapses, human capital accumulation allows the poorer households to increase their share of wealth, yet wealth inequality rises because the very low value of assets a_t disproportionally advantage households with a high enough share of wealth.

Over time, wealth inequality increases in both cases. However, it remains persistently lower under a moderate increase in patent breadth, reflecting the higher absolute asset values generated by a less aggressive strengthening of patent protection which tend to allow high enough wealth accumulation of the poorest households (given the rate of returns r_t), that is not possible when the asset value decreases in the presence of a low share of wealth as happens in the case of stronger patent protection.

Income inequality responds similarly to changes in patent protection. The mechanisms are made more evident when examining the income of an individual household κ:

$$\begin{aligned} I_t(\kappa) = r_t a_t(\kappa) + w_t(1-\tau) l_t(\kappa) + \psi(\kappa)\tau w_t L_t = \\ = r_t s_{a,t}(\kappa) a_t + (1-\tau) w_t h_t (s_{h,t}(\kappa) - u) + \psi(\kappa)\tau w_t L_t \end{aligned} \tag{49}$$

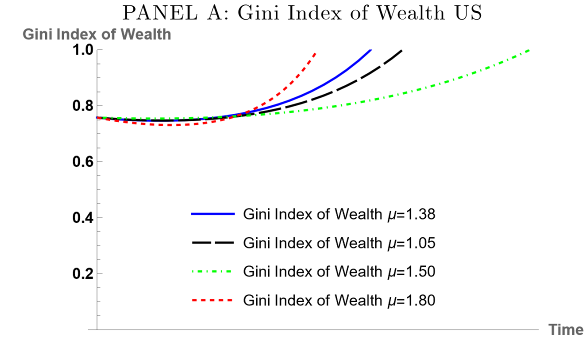
Households with greater asset holdings - i.e., higher levels of $a_t(\kappa)$ - experience a relatively higher income when the asset value increases due to a moderate increase in patent protection, and this positive effect is further reinforced when the household has a sufficiently large share of human capital at time t, $s_{h,t}(\kappa) > u$. Since wealthier households have larger share of assets $s_{a,t}(\kappa)$, and typically also hold a higher share of human capital, the combined operation of these mechanisms implies that a moderate increase in patent breadth disproportionately increases income among wealthier households. This interaction explains the positive effect of a modest strengthening of patent protection on income inequality. As time elapses, human capital accumulation magnifies income because households with a higher wealth share further increase their relative share. Figure 7, Panel B, illustrates this scenario for a moderate increase in patent protection, in which the average markup rises from its baseline value of $\mu = 1.38$ to $\mu = 1.50$. By contrast, when patent protection is tightened substantially - as shown in Figure 8, Panel B, where the markup increases from $\mu = 1.38$ to $\mu = 1.80$ - the mechanisms described above imply that the resulting decline in aggregate asset values, a_t, mitigates income inequality at least initially. Households with greater share of asset holdings - i.e., higher levels of $s_{a,t}(\kappa)$ - experience a relatively lower income when the asset value diaereses due to a tight patent protection, and this negative effect is only partially offset when the same household have a large enough share of human capital at time t, $s_{h,t}(\kappa) > u$. Since wealthier households have larger share of asset values $s_{a,t}(\kappa)$, and typically hold a higher share of human capital, the combined operation of these mechanisms implies that a large breadth disproportionately decreases income among richer households. This interaction explains the negative effect of a tight patent protection on income inequality.

As time elapses, human capital accumulation, which raises the share of human capital, together with a more sustained asset accumulation of the richer households increases income inequality. However, as for wealth inequality, income inequality remains persistently lower under a moderate increase in patent breadth, reflecting the higher absolute asset values generated by a less aggressive strengthening of patent protection which allow high enough wealth accumulation of the poorer households (given the rate of return on asset r_t), that is not possible when the asset value decreases in the presence of a low share of wealth, as in the case of stronger patent protection.

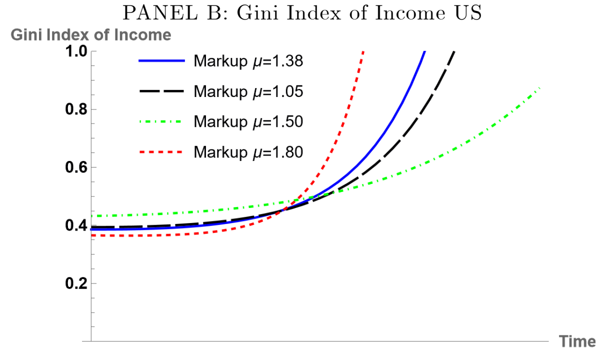
At the same time, the growth rate of the economy remains stable because it is mainly determined by the dynamics of human capital accumulation, which is not directly affected by the patent breadth.

Figure 7

PANEL A: Gini Index of Wealth US



PANEL B: Gini Index of Income US



The analysis and main results remain valid when patent breadth differs across sectors, generating heterogeneity in markups$\mu(i)$, quality $Z_t(i)$, firm size $X_t(i)$, profits $\Pi_t(i)$, market valuation $V_t(i)$, and R&D scale $L_t(i)$ (see Appendix H). Thus, the symmetric benchmark can be interpreted as a special case of an economy with sector-specific patent protection.

In this broader setting, the average markup summarizes the economy-wide degree of patent protection emerging from heterogeneous sectoral regimes. Because market size and R&D activity vary across industries, changes in patent breadth affecting only a subset of sectors alter the aggregate markup through composition effects. If stronger protection applies to relatively small sectors, the increase in the average degree of patent protection - and hence its general-equilibrium impact - is modest. By contrast, when large or economically central industries experience substantial strengthening of patent breadth, the average markup rises significantly, amplifying its effects on innovation, firm dynamics, and aggregate outcomes.

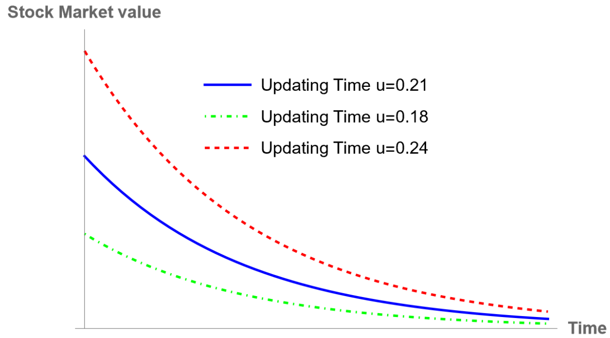
6.2 Change in updating

This section examines how time devoted to human capital updating shapes wealth and income inequality. An increase in the time allocated to updating human capital per unit of existing skills accelerates human capital accumulation and fosters the expansion of intermediate-good varieties, as described by equation (35). The entry of new intermediate producers spurs the demand for labor and the wage per unit of human capital, w_t, when the margin of profit is sufficiently high, i.e., inequality $\mu > 1 + (\beta\xi)/\alpha$ holds (see Appendix F). In this case, stronger human capital accumulation and the expansion of labor demand from new intermediate firms raise the wage bill associated with R&D activities, $w_t L_t(i)$, exerting downward pressure on the firms' expected market value, V_t.

However, the high enough profit margin μ induces intermediate firms to expand their quality-adjusted scale of production, x_t, in order to remain profitable. In this case, the joint effect of larger firm size and higher R&D labor costs - despite higher wages but fewer hours worked per firm - leads to an increase in firms' stock market valuation, V_t, provided that two conditions are satisfied. Specifically, firm value rises when the profit margin is large enough, that is, when $\mu > 1 + (\beta\xi)/\alpha$, and when the embodied human capital quality-adjusted size of the R&D laboratory is not excessively large, namely when $L_0(i)/Z_0 < \rho/\alpha$ (see Appendix F). Both conditions are increasing in the strength of technological spillovers, captured by a lower value of α, in both final-good production and intermediate-good demand. As shown by the production structure of the final good and the demand for intermediate inputs, a higher degree of spillovers implies that output and intermediate-good demand depend more strongly on economy-wide technological progress, Z_t, than on firm-specific innovations, $Z_t(i)$. When technological advances diffuse more broadly across firms, individual incumbents internalize a smaller fraction of the returns to their own R&D efforts. As a consequence, firms require sufficiently high profit margins to remain profitable, while a larger scale scale of R&D activity is needed to sustain firm-level technological advancement. This interaction highlights how stronger technological spillovers amplify the role of market structure in shaping firms' incentives to innovate and invest.

Figure 8 depicts this case.

Figure 8: Stock Market Value



The increase in firms' market valuations, together with the expansion in the mass of intermediate producers, raises the aggregate value of assets in the economy, $a_t = N_t V_t$. This expansion in aggregate asset values constitutes a key channel through which innovation-driven dynamics affect the distribution of wealth and income. Figures 9 depict the dynamic trajectories of the Gini coefficients for wealth (Panel A) and income (Panel B), respectively. The quantitative analysis shows that the condition $\mu > 1 + (\beta\rho)/\alpha$ is satisfied, implying that an increase in the updating interval leads to higher levels of both wealth and income inequality. Using an average markup of $\mu = 1.38$ as reported by Hall (2008), a discount rate of $\rho = 0.0331$ from the World Development Indicators, and a degree of technological spillovers of $(1 - \alpha) = 0.833$ from Iacopetta et al. (2019), the inequality condition continues to hold even when the parameter β is increased from its baseline value of 0.01 to 1.

To clarify the mechanisms driving wealth inequality, consider the law of motion for an individual household's wealth share in equation (48). The first term captures the effect of the household's wealth share relative to its consumption share, weighted by the ratio of aggregate consumption to aggregate asset value, c_t/a_t. As in the case of patent strength, this component governs how saving differentials translate into changes in relative asset positions. In particular, an increase in aggregate asset value a_t for instance, following a rise in updating effort - reduces the ratio c_t/a_t and thereby attenuates the relative advantage of wealthier households in accumulating assets. Conversely, a decline in a_t, as may occur when updating effort decreases, raises c_t/a_t and amplifies the asset accumulation advantage of households at the top of the wealth distribution.

Beyond this channel, a central mechanism shaping the time evolution of wealth inequality operates through the relative contribution of labor income and asset income to household wealth dynamics. When a household's human capital share is sufficiently large, $s_{h,t}(\kappa) > u$, labor income relative to aggregate asset value, $(w_t h_t)/a_t$, contributes positively to the evolution of its wealth share. An increase in updating effort raises both the wage flow per unit of human capital and the rate of human capital accumulation, thereby strengthening the positive contribution of labor earnings - relative to asset holdings - to wealth accumulation. At the same time, greater updating effort increases aggregate asset values a_t, which dampens the impact of labor income on individual wealth shares. Because wealthier households typically hold a larger share of assets relative to their share of human capital, the net effect is to reinforce wealth accumulation at the top of the distribution. As a result, higher updating effort amplifies wealth inequality relative to the baseline scenario.

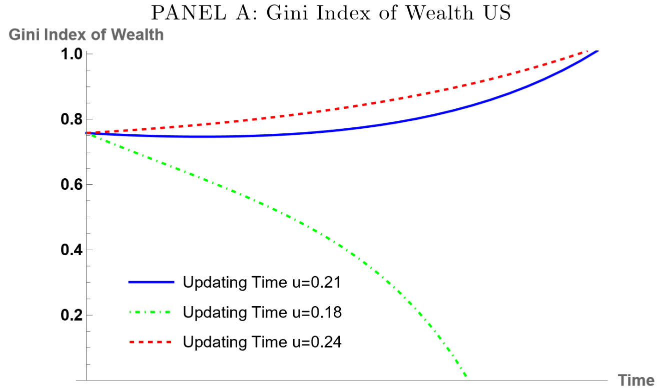
However, the dynamics of human capital accumulation introduce a countervailing force. The growth rate of individual human capital is inversely related to the household's share of aggregate human capital, $\dot{h}_t(\kappa)/h_t(\kappa) = [\xi(\kappa) u - \delta(\kappa)]/s_{h,t}(\kappa)$ so that households with a lower initial human capital share - typically poorer households - experience faster proportional growth in skills. Over time, this mechanism narrows human capital disparities and moderates the increase in wealth inequality. Consequently, although wealth inequality continues to rise, it does so at a slower pace than in the baseline case (see Figure 9, Panel A).

The same mechanisms govern the response of income inequality to changes in household updating effort. As shown in equation (49), household income consists of labor earnings and returns on asset holdings. An increase in updating effort raises wages, accelerates human capital accumulation, and boosts aggregate asset values. Because these components are initially more concentrated among wealthier households, their joint effect is to widen income disparities in the short run. However, the dynamics of human capital accumulation introduce an equalizing force. The growth rate of individual human capital is inversely related to the household's share of aggregate human capital, $\dot{h}_t(\kappa)/h_t(\kappa) = [\xi(\kappa)u - \delta(\kappa)]/s_{h,t}(\kappa)$ implying that households with a lower initial human capital share - typically poorer households - experience faster proportional skill growth. Over time, this convergence in human capital attenuates the increase in income inequality. As in the case of wealth inequality, income inequality continues to rise, yet it becomes lower than the baseline case in the long run (see Figure 9, Panel B).

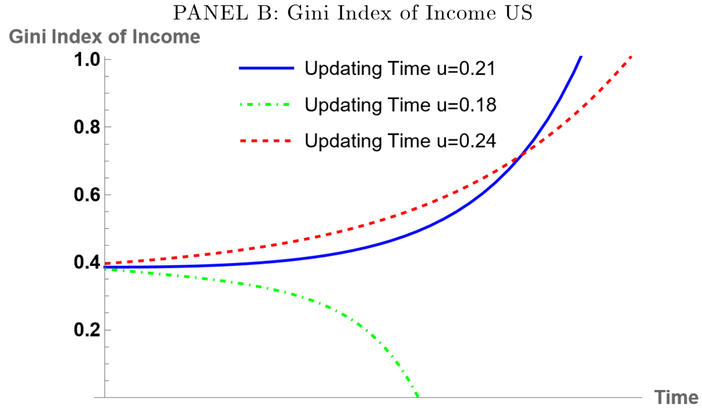
Finally, because GDP per capita growth is primarily driven by technological innovation and human capital accumulation - both of which respond positively to skill updating - an increase in the time devoted to updating skills raises the long-run growth rate of the economy and dampens the rise of wealth and income inequality. Therefore, an increase in the updating effort allows a negative relationship between both wealth and income inequality and growth of the GDP per capita to emerge in the long run.

Figure 9

PANEL A: Gini Index of Wealth US



PANEL B: Gini Index of Income US



7 Extensions

This section discusses several natural extensions of the baseline framework. As noted above, the analysis and main results continue to hold under an exogenous population growth process (see Appendix G).

A further extension concerns the specification of the household intertemporal budget constraint in equation (2). In the baseline formulation, firms remunerate workers at the prevailing wage per unit of human capital, w_t, only for effective labor time, $l_t(\kappa)$. More generally, one may allow firms to compensate both effective labor time and skill-updating time, u_t, at the same unit wage. This extension has a straightforward economic interpretation. In the model, individuals are employed exclusively in R&D activities, which are typically specialized and product-specific. A firm that compensates only effective labor time can therefore be interpreted as narrowly focused on its own technological domain. By contrast, a firm that finances a share of workers' updating effort engages in broader, more general R&D activities that extend beyond its immediate technological scope. A higher value of γ corresponds to a more wide-knowledge and less technologically specialized research strategy.

To simplify exposition, here we present the case where the updating effort is fully remunerated, i.e., $\gamma = 1$. The intertemporal budget constraint becomes

$$\dot{a}_t(\kappa) = r_t a_t(\kappa) + w_t(1-\tau)h_t(\kappa) + \psi(\kappa)\tau w_t h_t - c_t(\kappa), \quad (50)$$

All analytical steps carry over to this extended specification (see Appendices H and I). To conserve space, we focus here on the time paths of wealth and income inequality under broad-based remuneration of R&D effort, whereby firms compensate both effective labor time and updating time, $l_t(\kappa) + u_t(\kappa)$. We compare these outcomes with those obtained under the baseline specification, in which incumbent firms adopt a technologically specialized research strategy and remunerate only human capital embodied labor time, $l_t(\kappa)$.

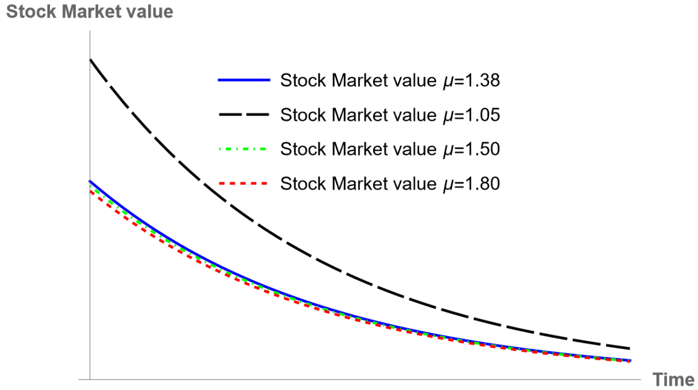
As in the benchmark case, the differential equation governing the evolution of household κ's share of aggregate wealth admits a closed-form analytical solution. However, because the explicit expression depends on a large set of structural parameters, we rely on numerical simulations to characterize the transitional dynamics. To ensure comparability with the previous analysis, the model is calibrated to aggregate U.S. data (see Appendix N for details).

The following sections examine the effects of patent strengthening and changes in updating effort on the dynamic evolution of wealth and income inequality, as well as on the growth rate of GDP per capita.

7.1 Strengthening patent protection

This section examines the effects of patent breadth on wealth and income inequality. The analysis carry over as delineated in the baseline version of the model set-up, yet the main difference that change some of the results lies in a higher cost firms incur for their R&D activity because they remunerate both the effective labor time and the updating effort of each individual. Particularly, when firms remunerate both production and updating time, the qualitative mechanisms identified in the baseline model continue to operate, though general equilibrium interactions partly differ. As in the benchmark case, an expansion in patent breadth raises the price of each intermediate variety and reduces productivity-adjusted firm size, x_t. The resulting compression of incumbents' market shares lowers entry costs and stimulates firm creation, generating offsetting effects on skilled labor demand: smaller incumbent scale depresses within-firm R&D employment, while entry expands aggregate research activity. However, because firms remunerate both the effective labor time $l_t(\kappa)$ and the updating effort $u_t(\kappa)$ of each worker, the raising cost of the wide-knowledge R&D activity reinforces the negative effect of the reduced firm size on its market valuation, and this result in lower stock market value of each incumbent firm. Therefore, a strengthening of patent protection, moderate and stronger, reduces the firm's market value V_t (see Figure 10)

Figure 10: Stock Market Value, wide-knowledge R&D



This monotonic effect of patent breadth on firms' asset valuation has non-monotonic effect on wealth and income inequality. To facilitate reading, the dynamic equation governing the time evolution of households' share of wealth is here reported

$$\begin{array}{c} \dot{s}_{a,t}(\kappa) = \frac{c_t}{a_t}\left(s_{a,t}(\kappa) - s_{c,t}(\kappa)\right) + \\ -\frac{w_t h_t}{a_t}\left[\left(s_{a,t}(\kappa) - \psi(\kappa)\tau\right) - (1-\tau)\, s_{h,t}(\kappa)\right], \end{array} \tag{51}$$

The first term in equation (L3) captures how a household's wealth share evolves as a function of the gap between its share of aggregate wealth and its share of aggregate consumption, scaled by the aggregate consumption-asset ratio. As in the baseline model, wealth accumulation depends on the relative weight of labor and asset income in total resources. For brevity, we do not repeat that analysis here.

A central mechanism driving the evolution of wealth inequality operates through the composition of income. An increase in patent breadth - whether moderate or strong - raises R&D costs, which take the form of higher wage payments to workers. At the same time, higher R&D expenditure lowers firms' expected stock market value and reduces the aggregate asset value a_t. As a result, the ratio of labor income to asset holdings, $(w_t h_t)/a_t$, increases. This shift strengthens the role of labor income in wealth accumulation and benefits households whose wealth share is relatively low compared to their human capital share.

The mechanism is reinforced by the dynamics of human capital accumulation. The growth rate of individual human capital is inversely related to the household's human capital share: households with a lower share experience faster growth. Formally, $\dot{h}_t(\kappa)/h_t(\kappa) = [\xi(\kappa)u - \delta(\kappa)]/s_{h,t}(\kappa)$, which is decreasing in $s_{h,t}(\kappa)$. Intuitively, agents further from the knowledge frontier enjoy higher marginal returns to skill accumulation. This accelerates wage growth precisely for those households that are initially less endowed with wealth, thereby compressing the wealth distribution over time.

Together, these channels explain why stronger patent protection - through higher markups and R&D intensity - can reduce wealth inequality in this framework. Figure 11, Panel A, illustrates this mechanism for moderate and strong increases in patent protection, corresponding to an increase in the average markup from its baseline value of $\mu = 1.38$ to $\mu = 1.50$ and to $\mu = 1.80$, respectively.

Income inequality responds in a similar manner. The underlying mechanisms become clearer when examining the income dynamics of an individual household κ:

$$\begin{aligned} I_t(\kappa) &= r_t a_t(\kappa) + w_t(1-\tau)h_t(\kappa) + \psi(\kappa)\tau w_t h_t = \\ &= r_t s_{a,t}(\kappa)a_t + (1-\tau)w_t h_t s_{h,t}(\kappa) + \psi(\kappa)\tau w_t h_t. \end{aligned} \tag{52}$$

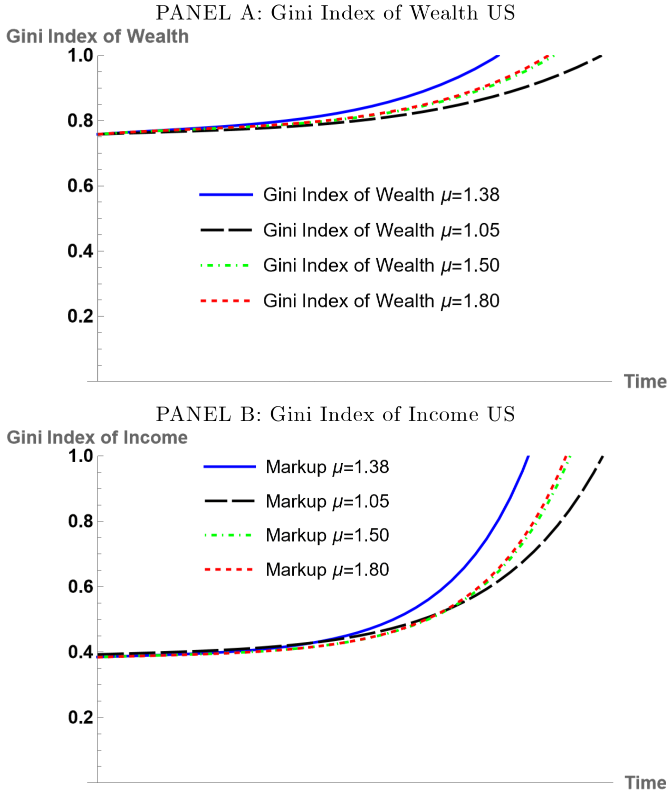
Wealthier households hold a larger share of aggregate assets, $s_{a,t}(\kappa)$, and therefore derive a greater fraction of their income from capital. When patent breadth increases - either moderately or strongly - firms' expected stock market value declines because higher R&D expenditures reduce net profits. This lowers the aggregate asset value a_t and weakens the capital-income channel. As a consequence, households whose wealth share is small relative to their human capital share benefit disproportionately, since labor income becomes more important in total income determination.

This redistributive effect is amplified by the dynamics of human capital accumulation. The growth rate of individual human capital is inversely related to the household's share of aggregate human capital: households with a lower share experience faster growth. Economically, agents that are further from the knowledge frontier enjoy higher marginal returns to skill accumulation. This accelerates wage growth at the lower end of the distribution, further compressing income disparities. These mechanisms explain why stronger patent protection reduces income inequality in the model. Figure 11, Panel B, illustrates this result for moderate and strong increases in patent protection, corresponding to a rise in the average markup from its baseline value of $\mu = 1.38$ to $\mu = 1.50$ and to $\mu = 1.80$, respectively.

At the same time, the aggregate growth rate remains essentially unchanged. Long-run growth is primarily driven by the accumulation of human capital, and patent breadth does not directly alter the fundamental parameters governing its evolution. As a result, the economy experiences a redistribution of income without a significant change in its growth trajectory.

Figure 11

PANEL A: Gini Index of Wealth US

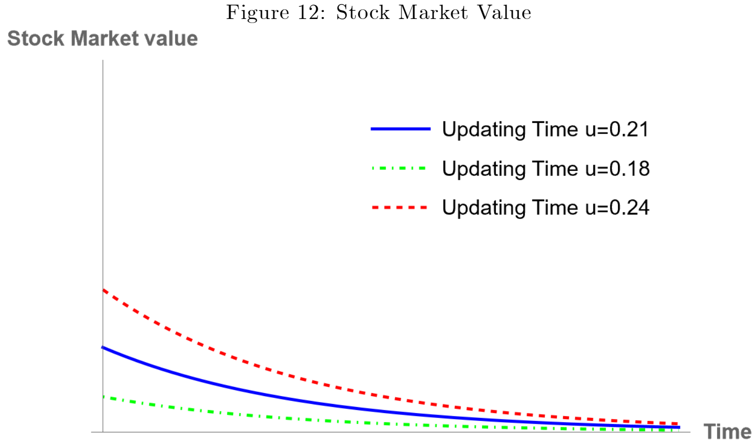


7.2 Change in updating

This section analyzes how time allocated to human capital updating affects wealth and income inequality. A higher updating intensity per unit of existing skills accelerates human capital accumulation, stimulates the expansion of intermediate-good varieties (35), and increases the wage per unit of human capital, w_t. Entry of new intermediate producers reduces R&D labor per firm, lowering unit labor costs initially. Over time, however, aggregate labor costs rise at the human capital growth rate, g_h. Stronger human capital accumulation therefore increases each firm's R&D wage bill, $w_t h_t(i)$, exerting downward pressure on expected firm value, V_t.

If the profit margin μ is sufficiently high, the increase in fixed operating costs associated with more advanced technologies, σZ_t, induces firms to expand their quality-adjusted scale, x_t, to preserve profitability. Firm value rises when two conditions hold: $\mu > 1 + (2\beta\xi)/\alpha$ and $L_0(i)/Z_0 < 2\xi/\alpha$ (see Appendix M). Both thresholds increase with the strength of technological spillovers (lower α) in final-good production and intermediate demand. Greater spillovers raise the dependence of output and demand on aggregate technology Z_t relative to firm-specific innovation $Z_t(i)$, reducing the appropriable returns to R&D. Firms therefore require higher markups and appropriately scaled R&D to sustain innovation. Stronger spillovers thus magnify the role of market structure in shaping innovative incentives (Figure 12).

Figure 12: Stock Market Value



The increase in firms' market valuations, together with the expansion in the mass of intermediate producers, raises the aggregate value of assets in the economy, $a_t = N_t V_t$. This expansion in aggregate asset values constitutes a key channel through which innovation-driven dynamics affect the distribution of wealth and income. Figures 13 depict the dynamic trajectories of the Gini coefficients for wealth (Panel A) and income (Panel B), respectively.

To clarify the mechanisms driving wealth inequality, consider the law of motion for an individual household's wealth share in equation (L3). As indicated above, the first term in equation (L3) captures the effect of the household's wealth share relative to its consumption share, weighted by the ratio of aggregate consumption to aggregate asset value, c_t/a_t. As in the case of patent strength, this component governs how saving differentials translate into changes in relative asset positions. In particular, an increase in aggregate asset value a_t for instance, following a rise in updating effort - reduces the ratio c_t/a_t and thereby attenuates the relative advantage of wealthier households in accumulating assets. Conversely, a decline in a_t, as may occur when updating effort decreases, raises c_t/a_t and amplifies the asset accumulation advantage of households at the top of the wealth distribution.

Beyond the direct effects discussed above, the evolution of wealth inequality is shaped by the relative contribution of labor and asset income to household wealth dynamics. When a household's share of aggregate human capital, $s_{h,t}(\kappa)$, is sufficiently large, labor income relative to aggregate asset value, $(w_t h_t)/a_t$, exerts a positive effect on its wealth share. An increase in updating effort raises both the wage per efficiency unit of labor and the rate of human capital accumulation, thereby strengthening the role of labor earnings - relative to asset income - in wealth formation.

At the same time, greater updating effort increases aggregate asset values a_t, which dampens the marginal impact of labor income on individual wealth shares. Since wealthier households typically hold a larger share of financial assets relative to their human capital share, this channel tends to reinforce wealth accumulation at the top of the distribution.

However, the mechanism differs from the case where firms are narrowly focused on its own technological domain because workers are compensated not only for effective labor time, $l_t(\kappa)$, but also for time devoted to updating effort, $u_t(\kappa)$. Hence, total labor income reflects both productive work and skill investment. This distinction is crucial. Under the human-capital-adjusted time constraint, $1 = l_t(\kappa)/h_t(\kappa) + u/s_{h,t}(\kappa)$, an increase in updating effort u

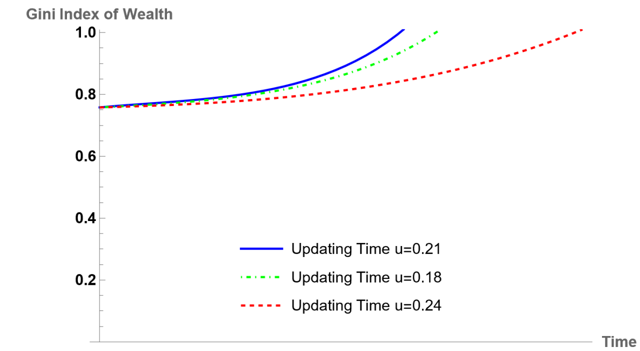
would reduce effective labor time - and thus wage income - if firms paid only for $l_t(\kappa)$, for annoy given level of the household's human capital, $h_t(\kappa)$ and of its share $s_{h,t}(\kappa)$. In contrast, when firms compensate both effective labor and updating effort, higher skill investment raises total labor earnings. As a result, updating effort mitigates the increase in wealth inequality relative to the baseline scenario. In addition, the growth rate of individual human capital is inversely related to the household's share of aggregate human capital, $\dot{h}_t(\kappa)/h_t(\kappa) = [\xi(\kappa)u - \delta(\kappa)]/s_{h,t}(\kappa)$. Households with a lower initial human capital share - typically poorer households - therefore experience faster proportional skill growth. Over time, this convergence mechanism narrows human capital disparities and moderates the rise in wealth inequality. Consequently, although wealth inequality continues to increase, it does so more slowly than in the baseline case (see Figure 13, Panel A).

The response of income inequality to changes in household updating effort is governed by analogous mechanisms. As shown in equation (52), individual income consists of labor earnings and returns on asset holdings. An increase in updating effort raises wages by enhancing effective labor supply, accelerates human capital accumulation, and increases aggregate asset values. Since both human capital and assets and are initially more concentrated among wealthier households, these effects tend to amplify income inequality in the short run. However, the dynamics of human capital accumulation generate an offsetting force. The growth rate of individual human capital is inversely related to the household's share of aggregate human capital. Households with a lower initial human capital share - typically poorer households - therefore experience faster proportional skill growth. This convergence mechanism gradually compresses wage differentials and attenuates the initial rise in income inequality. As in the case of wealth inequality, income inequality may continue to increase, but it eventually falls below its baseline level in the long run (Figure 13, Panel B).

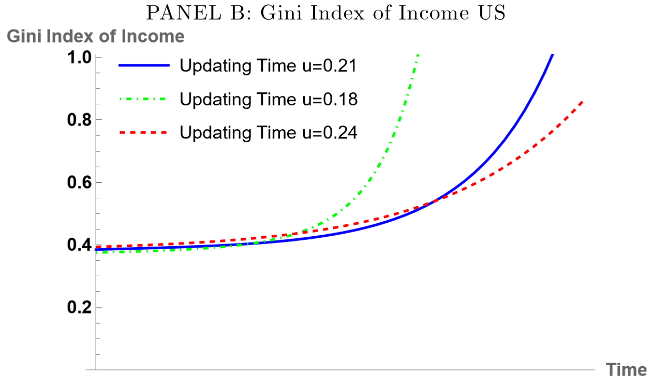
Finally, GDP per capita growth is primarily driven by technological innovation and human capital accumulation, which respond positively to greater skill updating. An increase in time devoted to updating effort therefore raises the long-run growth rate of the economy. Because the equalizing effects of human capital convergence materialize over time, the model implies a negative relationship between inequality - both wealth and income - and long-run GDP per capita growth.

Figure 13

PANEL A: Gini Index of Wealth US



PANEL B: Gini Index of Income US



8 Conclusions

Innovation has long been recognized as a central driver of economic growth, yet its distributional consequences remain the subject of ongoing debate. This paper contributes to that debate by developing an innovation-driven growth model with quality-improving and variety-expanding innovations in which heterogenous human capital accumulation and continuous skill updating are essential prerequisites for effective R&D activity. By linking asset accumulation and wage income to endogenous human capital formation in an economy with technological change and endogenous market structure, the model successfully accounts for key features of U.S. inequality dynamics over recent decades.

A central result is that patent policy has non-monotonic distributional effects operating through firms' valuations and the relative importance of labor and asset income. The non-monotonic effect of patent breadth on asset valuation extends to wealth and income inequality. A moderate strengthening of patent protection increases aggregate asset values but reduces the relative contribution of labor income to wealth accumulation, thereby conferring a short-run advantage on wealthier households. Over time, however, human capital accumulation enables poorer households to expand their wealth share, mitigating the rise in inequality. By contrast, a substantial tightening of patent protection lowers aggregate asset values and initially compresses wealth inequality. As time progresses, although human capital accumulation allows poorer households to increase their wealth share, the depressed level of asset values disproportionately benefits households with an initially higher wealth share, leading to a subsequent increase in inequality. In the long run, inequality rises under both regimes, but it remains persistently lower following a moderate expansion of patent breadth. Income inequality responds similarly to changes in patent protection. Higher asset valuations under a less aggressive strengthening of patent protection sustain sufficient wealth accumulation and income among poorer households - an outcome that does not obtain when asset values decline sharply under stronger protection.

This paper also analyzes how skill updating shapes innovation, market structure, and inequality. Allocating more time to human capital accumulation accelerates skill growth and expands the range of intermediate varieties, raising labor demand and wages. Firms respond by scaling up production. When profit margins are sufficiently high and R&D activities remain limited in size, larger scale and higher expected productivity increase firm valuations - an effect reinforced by stronger technological spillovers. Entry and higher valuations, in turn, raise aggregate asset holdings. The distributional effects are initially unequal. Because wages, human capital, and asset values are more concentrated among wealthier households, greater updating effort widens income and wealth inequality in the short run. Over time, however, human capital dynamics generate convergence: households with low initial skill shares experience faster proportional growth, moderating disparities. Consequently, inequality rises only modestly in the long run.

Since per capita growth is driven by innovation and human capital accumulation - both strengthened by skill updating - greater updating effort increases long-run growth while dampening persistent inequality. The model thus predicts a negative long-run relationship between wealth and income inequality and GDP per capita growth.

Appendix A

This Appendix obtains the optimal choices of a typical household κ. Each consumer maximizes eq. (1) choosing the optimal plan of $c_t(\kappa)$, l_t, subject to the law of motion of the asset equation and human capital as in equations (2) and (6) respectively, and to the time constraint as in eq. (5). The current value Hamiltonian is

$$\begin{aligned} H_t = \ln\left(c_t(\kappa)\right) + \\ +\lambda_{at}\left[r_t a_t(\kappa) + (1-\tau) w_t l_t(\kappa) + \psi(\kappa)\tau w_t L_t - c_t(\kappa)\right] + \\ +\lambda_{ht}\left[\xi(\kappa)\left(h_t(\kappa) - l_t(\kappa)\right) - \delta(\kappa) h_t\right]. \end{aligned} \tag{A1}$$

The optimal conditions are:

$$\frac{\partial H_t}{\partial c_t(\kappa)} = 0 \Rightarrow \frac{1}{c_t(\kappa)} = \lambda_{at}, \tag{A2}$$

$$\frac{\partial H_t}{\partial l_t(\kappa)} = 0 \Rightarrow \lambda_{at}(1-\tau) w_t - \lambda_{ht}\xi(\kappa) = 0, \tag{A3}$$

$$-\frac{\partial H_t}{\partial a_t(\kappa)} = \dot{\lambda}_{at} - \rho\lambda_{at} = -\lambda_{at} r_t, \tag{A4}$$

$$-\frac{\partial H_t}{\partial h_t(\kappa)} = \dot{\lambda}_{ht} - \rho\lambda_{ht} = -\lambda_{ht}\xi(\kappa). \tag{A5}$$

From conditions (A4) and (A5), the law of motion of the costate variables λ_{at} and λ_{ht} are respectively $\left(\dot{\lambda}_{at}/\lambda_{at}\right) = \rho - r_t$ and $\left(\dot{\lambda}_{ht}/\lambda_{ht}\right) = \rho - \xi(\kappa)$. From condition (A3) we obtain $\left(\dot{\lambda}_{at}/\lambda_{at}\right) + \left(\dot{w}_t/w_t\right) = \left(\dot{\lambda}_{ht}/\lambda_{ht}\right)$, that, using the law of motion of the costate variables λ_{at}, i.e., $\left(\dot{\lambda}_{at}/\lambda_{at}\right) = \rho - r_t$, can be written as: $\left(\dot{w}_t/w_t\right) = r_t - \xi(\kappa)$. Using again the law of motion of the costate variables λ_{at}, from condition (A2), the standard law of motion of the per capita consumption becomes $\dot{c}_t(\kappa)/c_t(\kappa) = r_t - \rho$.

Using the law of motion of human capital of a household κ (6), the aggregate human capital accumulation of the economy is

$$\begin{aligned} \dot{h}_t = \int_0^1 \dot{h}_t(\kappa)\, dF(\kappa) = \\ = h_t \int_0^1 \left(\xi(\kappa) u - \delta(\kappa)\right) dF(\kappa) = h_t(\xi u - \delta), \end{aligned} \tag{A6}$$

where $\xi = \int_0^1 \xi(\kappa)\, dF(\kappa)$, $\delta = \int_0^1 \delta(\kappa)\, dF(\kappa)$. Therefore, the growth rate of human capital becomes:

$$g_h \equiv \frac{\dot{h}_t}{h_t} = \xi u - \delta. \tag{A7}$$

Q.E.D.

Appendix B

The current-value Hamiltonian for monopolistic firm i is given by (15). To introduce the upper bound μ on price $p_t(i)$, we modify (15) as follows

$$H_t(i) = \Pi_t(i) - w_t L_t(i) + \eta_t \dot{Z}_t(i) + \omega_t(i)\,(\mu - p_t(i)), \tag{B1}$$

where ω_t is the multiplier on $p_t(i) \leq \mu$. Substituting (11), (12) and (13 into (B1), we can derive the optimal conditions

$$\frac{\partial H_t(i)}{\partial p_t(i)} = 0 \Rightarrow \frac{\partial \Pi_t(i)}{\partial p_t(i)} = \omega_t(i), \tag{B2}$$

$$\frac{\partial H_t(i)}{\partial L_t(i)} = 0 \Rightarrow w_t = \eta_t, \tag{B3}$$

$$\begin{gathered} \tfrac{\partial H_t(i)}{\partial Z_t(i)} = 0 \Rightarrow \\ \alpha\left[(p_t(i) - 1)\left(\tfrac{\theta}{p_t(i)}\right)^{\frac{1}{1-\theta}} \tfrac{E_t}{N_t}\right] Z_t^{\alpha-1}(i)\, Z^{1-\alpha} = r_t \eta_t - \dot{\eta}_t. \end{gathered} \tag{B4}$$

If $p_t(i) < \mu$, then $\omega_t(i) = 0$. In this case, $\partial \Pi_t(i) / \partial p_t(i) = 0$ yields $p_t(i) = 1/\theta$. If the constraint on $p_t(i)$ is binding, then $\omega_t(i) > 0$. In this case, we have $p_t(i) = \mu$, proving (16). Given that we assume $\mu < 1/\theta$, $p_t(i) = \mu$ always holds.

Using condition (B3) to substitute w_t for η_t into (B4), and using $r_t = (\dot{w}_t / w_t) + \xi(\kappa)$ to substitute for r_t, condition (B4) can be written as:

$$\begin{aligned} \alpha\left[(\mu - 1)\left(\tfrac{\theta}{\mu}\right)^{\frac{1}{1-\theta}} \tfrac{E_t}{N_t}\right] Z_t^{\alpha-1}(i)\, Z_t^{1-\alpha} &= \\ = \left(\tfrac{\dot{w}_t}{w_t} + \xi(\kappa)\right) w_t - \dot{w}_t&, \end{aligned} \tag{B5}$$

where $p_t(i) = \mu$ is used. Assuming symmetry, i.e. $Z_t(i) = Z_t$, (B5) reduces to

$$\alpha(\mu - 1)\left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}} \frac{E_t}{N_t} = \xi(\kappa)\, w_t. \tag{B6}$$

From (B6), we can derive the wage flow:

$$\begin{aligned} w_t &= \tfrac{\alpha}{\xi(\kappa)} \left(\mu - 1\right) \left(\tfrac{\theta}{\mu}\right)^{\frac{1}{1-\theta}} \tfrac{E_t}{N_t} = \\ &= \tfrac{\alpha}{\xi(\kappa)} \left(\mu - 1\right) \left(\tfrac{1}{\mu}\right)^{\frac{1}{1-\theta}} x_t, \end{aligned} \tag{B7}$$

where in the last equality on the right hand side (18) has been used. Equation (B7) indicates that the growth rate of the quality-adjusted firm size, x_t, coincides with the growth rate of the wage flow, i.e., $(\dot{x}_t/x_t) = (\dot{w}_t/w_t) = r_t - \xi(\kappa)$. Q.E.D.

Appendix C

This appendix derives the expression for the quality-adjusted firm size, x_t, establishes its stationarity, and identifies the parameter restrictions that ensure x_t remains strictly positive. Using the growth rate of aggregate quality, $z_t = \left(\dot{Z}_t/Z_t\right)$, and substituting equation (B7) from Appendix B for the wage flow w_t, together with the condition $(\dot{x}_t/x_t) = r_t - \xi(\kappa)$, equation (22) can be rewritten as:

$$\begin{aligned} r_t &= \frac{(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t - \sigma Z_t}{\beta\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t} + \\ &- \frac{\frac{\alpha}{\xi(\kappa)}\left[(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} x_t\right] L_t}{\beta\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t} + z_t + r_t - \xi(\kappa). \end{aligned} \tag{C1}$$

Using $\dot{Z}_t = L_t$, the expression for the quality-adjusted firm size x_t, as given in condition (C1), can be equivalently rewritten as:

$$\begin{aligned} x_t &= \frac{\sigma}{\beta\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}} \left[\frac{\mu-1}{\beta} + z_t\left(1 - \frac{\alpha}{\xi(\kappa)}\frac{\mu-1}{\beta}\right) - \xi(\kappa)\right]^{-1} = \\ &= \frac{\sigma}{\beta}\left(\mu\right)^{\frac{1}{1-\theta}} \left[\frac{(\mu-1)}{\beta} + z_t\left(1 - \frac{\alpha}{\xi(\kappa)}\frac{\mu-1}{\beta}\right) - \xi(\kappa)\right]^{-1}. \end{aligned} \tag{C2}$$

Because z_t is stationary and equals $z_t = \left(\dot{h}_t/h_t\right) = g_h = \xi u - \delta$, condition (C2) implies the stationarity of the quality-adjusted firm size, x_t, which, considering equation (B7), implies the stationarity of the wage flow, w_t, i.e., $\dot{w}_t/w_t = r_t - \xi(\kappa) = 0$. This implies that $r_t = \xi(\kappa) = \xi$, and by the Euler equation for consumption $\dot{c}_t(\kappa)/c_t(\kappa) = r_t - \rho$, we have $\dot{c}_t(\kappa)/c_t(\kappa) = \dot{c}_t/c_t = r_t - \rho = \xi - \rho$.

The first term on the right hand side of condition (C2), $(\mu - 1)/\beta$, is positive. The second term in the square brackets of the right hand side of condition (C2) is positive when inequality

$$\frac{(\mu - 1)}{\beta}\left(1 - \frac{\alpha}{\xi} z_t\right) + (z_t - \xi) > 0 \tag{C3}$$

holds. The first term of condition (C3) is positive whenever $1 - z_t(\alpha/\rho) > 0$ holds. This inequality is true for $u < \delta/\xi + 1/\alpha$. The second term of condition (C3) have to be strictly negative„ i.e.,. $(z_t - \rho) < 0$ must hold. Indeed, for $(z_t - \rho) > 0$, the market value of the monopolistic firm, V_t, diverges, i.e., $V_t \to \infty$. This result follows directly from the expression for the monopolistic firm's market value given in (29), where the positive discounting term $(z_t - \rho) > 0$ would render the present value of future profits unbounded, i.e.,

$$\begin{aligned} V_t = {} & \left[\exp(z_t - \xi) \bullet \infty - \exp(z_t - \xi) \bullet t\right] \times \\ & \times \frac{\left[\left((\mu-1)\left(\frac{1}{\mu}\right)^{(1/(1-\theta))} x_t - \sigma\right) Z_0 - w_t L_0(i)\right]}{z_t - \xi}, \end{aligned} \tag{C4}$$

which diverges.

Therefore, the only internally consistent case is characterized by $(\xi - z_t) > 0$, that always holds because $\xi - z_t = \xi(1 - u) + \delta > 0$ holds. As a consequence, ensuring that the quality-adjusted firm size x_t remains strictly positive requires the imposition of a suitable parameter restriction. To formalize this, we rewrite the term of condition (C3), as follows:

$$\frac{(\mu - 1)}{\beta}\left(1 - z_t \frac{\alpha}{\xi}\right) - (\xi - z_t) > 0, \tag{C5}$$

where inequalities $1 - z_t(\alpha/\rho) > 0$ and $(\rho - z_t) > 0$ hold. Consequently, the following parameter restriction ensures a strictly positive value for the quality-adjusted firm size x_t:

$$1 > \frac{\beta\rho}{\mu - 1} + z_t\left(\frac{\alpha}{\xi} - \frac{\beta}{\mu - 1}\right), \tag{C6}$$

that can be rewritten as

$$u < \frac{\frac{\mu-1}{\beta}\left(1 - \frac{\delta}{\xi}\right) - (\delta + \xi)}{\alpha\frac{\mu-1}{\beta} - \xi} \tag{C7}$$

The parameter restrictions obtained above all hold when the following sufficient condition holds

$$u < Min\left\{\frac{\delta}{\xi}+\frac{1}{\alpha},\frac{\frac{\mu-1}{\beta}\left(1-\frac{\delta}{\xi}\right)-(\delta+\xi)}{\alpha\frac{\mu-1}{\beta}-\xi}\right\}. \tag{C8}$$

Q.E.D.

Appendix D

Income received by household κ is given by

$$\begin{aligned} I_t(\kappa) &= r_t a_t(\kappa) + (1-\tau) w_t l_t(\kappa) + \psi(\kappa)\tau w_t L_t = \\ &= r_t a_t s_{a,t}(\kappa) + (1-\tau) w_t l_t + \psi(\kappa)\tau w_t L_t, \end{aligned} \tag{D1}$$

where the identity index κ is uniformly distributed between 0 and 1. We now order the households in an ascending order of income. The Gini coefficient of income is given by $\sigma_I = 1 - 2b_I$, where

$$b_I = \int_0^1 \Lambda_I(\kappa)\, d\kappa. \tag{D2}$$

The Lorenz curve $\Lambda_I(\kappa)$ of income is given by

$$\begin{aligned} \Lambda_I(\kappa) &\equiv \frac{\int_0^\kappa I(\chi)d\chi}{\int_0^1 I(\chi)d\chi} = \\ &= \frac{r_t a_t \int_0^\kappa s_{a,t}(\chi)d\chi + (1-\tau)w_t \int_0^\kappa l_t(\chi)d\chi + \tau w_t L_t \int_0^\kappa \psi(\chi)d\chi}{r_t a_t + w_t L_t}, \end{aligned} \tag{D3}$$

where

$$\begin{aligned} \int_0^\kappa l_t(\chi)\, d\chi &= \int_0^\kappa \left(h_t(\chi)(h_t/h_t) - u h_t\right) d\chi = \\ &= h_t\left((1/2)\, s_{h,t}^2(\kappa) - u\kappa\right), \end{aligned} \tag{D4}$$

$\int_0^\kappa \psi(\chi)\, d\chi = (1/2)\,\psi^2(\kappa)$, and $\int_0^\kappa s_{a,t}(\chi)\, d\chi$ is the Lorenz curve $\Lambda_a(\kappa)$ of wealth. To see this,

$$\Lambda_a(\kappa) \equiv \frac{\int_0^\kappa a(\chi)\, d\chi}{\int_0^1 a(\chi)\, d\chi} = \frac{\int_0^\kappa a(\chi)\, d\chi}{a} = \int_0^\kappa s_{a,t}(\chi)\, d\chi. \tag{D5}$$

Substituting (D3) and (D5) into (D2), yields

$$b_I = \frac{r_t a_t}{r_t a_t + w_t L_t} \int_0^1 \Lambda_a(\kappa)\, d\kappa + \\ + \frac{(1-\tau) w_t h_t \frac{1}{2} \int_0^1 \left[s_{h,t}^2(\kappa) - u\kappa \right] d\kappa + \tau w_t h_t (1-u) \frac{1}{2} \displaystyle\int_0^1 \psi^2(\kappa) d\kappa}{r_t a_t + w_t L_t}, \tag{D6}$$

where $\int_0^1 \Lambda_a(\kappa)\, d\kappa \equiv b_a$, and we have used $L_t = h_t(1-u)$ to substitute for L_t. Recall that the Gini coefficient of wealth is given by $\sigma_a = 1 - 2b_a$. Therefore, substituting (D6) into $\sigma_I = 1 - 2b_I$ yields the Gini coefficient of income given by

$$\sigma_{I,t} = \frac{r_t a_t}{r_t a_t + w_t h_t (1-u)} \sigma_{a,t} + \\ + \frac{w_t h_t}{r_t a_t + w_t h_t (1-u)} \left[(1-u)(1-\tau)\left(\frac{1}{3} - \frac{1}{2}u \right) + \frac{1}{3}\tau (1-u) \right], \tag{D7}$$

where $\sigma_{a,t}$ is the Gini coefficient of wealth. Q.E.D.

Appendix E

This appendix provides an explanation of how the parameter values are calibrated. Table 1 shows some key parameter values with the respective source used.

Table 1: Baseline Calibration US

$r_t = 0.0331$	WDI (2023)
$\mu = 1.38$	Hall (2018)
$s_{ct} = 0.35$	Attanasio and Pistaferri (2016)
$u = 0.21$	Tenopir et al.,2012
$\tau = 0.106$	Cathrerine et al. (2025)
$\psi(\kappa) = 0.215$	CBO (2024)
$1 - \alpha = 0.833$	Iacopetta et al. (2019)
$g_h = 0.004$	Mulligan and Sala-i-Martin (2000)
$g = 1.8\%$	WDI (2023)
$\delta = 0.01$	calibrated
$\rho = 0.0151$	calibrated
$\phi = -0.714$	calibrated
$\theta = 0.5$	calibrated
$\beta = 0.01$	calibrated
$\sigma = 0.194$	calibrated

The interest rate, denoted by r_t, is set to the conventional value of 0.0331, which implies a real interest rate of 3.31 percent. This figure is consistent with the long-term average real interest rate for the period 2000-2021, as reported by the OECD Statistics database (annual percentage). It is important to emphasize that alternative specifications of r_t do not qualitatively alter the calibration results.

Hall (2018) provides estimates of price markups for the US economy over the period 1988-2015, reporting values ranging from 1.04 to 1.85, with an average markup of $\mu = 1.38$. The highest estimated markup, $\mu = 1.85$, pertains to the Agriculture, Forestry, Fishing, and Hunting sectors, while the second-highest, $\mu = 1.55$, corresponds to the Accommodation and Food Services sector. The parameter θ is calibrated to satisfy the inequality $\mu < 1/\theta$, where μ represents the markup. Following Hall (2018), we adopt the highest estimated markup value, $\mu = 1.85$, corresponding to the Accommodation and Food Services sector. Imposing the condition $\theta < 1/\mu$ yields $\theta < (1/1.85) \simeq 0.54$. Accordingly, we set $\theta = 0.5$, a value that satisfies this inequality and ensures internal consistency within the model.

The parameter governing the share of consumption, denoted by $s_{c,t}(\kappa)$, is calibrated following Attanasio and Pistaferri (2016). In particular, Figure 1 of Attanasio and Pistaferri (2016) documents the evolution of consumption inequality over time and across empirical studies. Consumption inequality is measured as the variance of the logarithm of per capita consumption, adjusted for inflation using the Consumer Price Index. The reported values of this measure range from 0.2 to 0.45. We adopt an average value of $s_{c,t}(\kappa) = 0.35$. As with the discount rate, sensitivity analysis indicates that alternative values of $s_{c,t}(\kappa)$ do not qualitatively affect the model's implications for the dynamic behavior of wealth and income.

While a growing body of empirical work underscores the crucial role of human-capital updating in enhancing absorptive capacity and mediating the effects of R&D on productivity and innovation, to date, there is no clean, widely cited empirical economics paper that quantifies hours of formal training by researchers and ties that precisely to innovation outcomes. Survey-based estimates (Tenopir et al., 2012) report annual hours spent reading: averages reported in the literature range around 300-450 hours per year on scholarly reading (e.g., $\sim$22 articles/month $\times$ $\sim$0.5-0.8 hours/article, which implies about 216 hours/year for articles alone; totals including books/other is 300–450 hours/year). Considering a working time of 8 hours per day, 22 working days per month, the total amount of working hours per year is approximately 2112. Given the updating working time of approximately 450 hours per year and 2112 working hours per year, the share of the updating time per year is around 0.21. We use this evidence to calibrate the fraction of time allocated by individuals to human capital updating at $u = 0.21$. The same calibrated value can obtained considering that the average academic staff member spends over 300 hours each year on journal article readings, 180 hours on book readings, and around 120 hours on other publication readings, for a total commitment of 76 eight-hour days each year, with 360 days for each year (Tenopri et al., 2012).

The tax rate used to finance the Social Security system is taken from Catherine et al. (2025) and is set to $\tau = 0.106$. The parameter $\psi(\kappa)$, which captures the distribution of Social Security assets across households, is calibrated using data from the Congressional Budget Office (CBO, 2024). Specifically, we compute the weighted average over the period 1989-2019 of mean Social Security asset holdings (expressed in thousands of 2022 dollars) among households that hold such assets, across different segments of the wealth distribution: the 25th percentile, the 26th–50th percentiles, the 51st-90th percentiles, and the top 10 percent. This procedure yields a calibrated value of $\psi(\kappa) = 0.215$, which is used in the individual wealth dynamics equation (42).

To calibrate the distribution of Social Security assets across the wealth distribution, we rely on data from the Congressional Budget Office (CBO, 2024). Over the period 1989-2019, the share of Social Security assets held by households in the top 10 percent of the wealth distribution increased modestly, from 7 to 8 percent. In contrast, the shares held by households in the bottom 25 percent and by those between the 26th and 50th percentiles rose substantially, from 31 to 49 percent and from 42 to 49 percent, respectively. The quantitative results of the model are robust to whether the initial or final asset shares over this period are used in the calibration. In addition, we calibrate the distributional parameters using information on the average value of Social Security assets held by families within each wealth group. Specifically, we use the mean values (expressed in thousands of 2022 dollars) for households at the 25th percentile, the 50th percentile, and the top 10 percent of the wealth distribution, averaged over the period 1989-2019. These data are also taken from the Congressional Budget Office (CBO, 2024). Based on these moments, the parameter $\tilde{\psi}$ in equation (D7) is set equal to 0.035 for households at the 25th percentile of the wealth distribution, 0.116 for households at the 50th percentile, and 0.54 for households in the top 10 percent. The model's quantitative implications are robust to this alternative calibration strategy, confirming that the main results do not depend sensitively on the specific moments used to discipline the distribution of Social Security asset holdings.

The initial values of human capital-embodied labor supply (l_0), per capita consumption (c_0), and wage flow (w_0) are all normalized to unity: $l_0 = c_0 = w_0 = 1$. We follow Iacopetta et al. (2019) to set the degree of technology spillovers $(1 - \alpha)$ to 0.833.

We calibrate the depreciation rate of human capital δ using the model-implied growth rate of human capital, $g_h = (\xi u - \delta)$, targeting its empirical counterpart. Following Mulligan and Sala-i-Martin (2000), we set the average annual growth rate of human capital in the U.S. economy to 0.4 percent. The productivity parameter governing human capital accumulation, ξ, is equal to the interest rate, i.e., $\xi = r_t = 0.0331$. Then the value of the the depreciation rate of human capital $\delta = 0.002951$. The congestion parameter in the production of new intermediate varieties, ϕ, is calibrated to ensure that the model-implied growth rate of GDP per capita, $g = (\xi u - \delta)(2 + \phi) / (1 + \phi)$ matches its empirical counterpart. According to the World Development Indicators (WDI, 2023), the average growth rate of U.S. GDP per capita over the period 1976-2022 is 1.8 percent, which the model reproduces under the chosen parameterization. The implied value of the congestion parameter is $\phi = -0.714$. The subjective discount rate ρ is calibrated using the Euler equation for consumption $\dot{c}_t/c_t = r_t - \rho$. Since the model specification implies that the growth rate of the consumption per capita is equal to the growth rate of the GDP per capita $g = 0.018$, the value of the subjective discount rate is set at $\rho = 0.0151$.

Conditional on the parameter values discussed above, the remaining unknowns in the equilibrium conditions are β and σ. These parameters are calibrated using the firm-size condition (27) and the wage equation (28). Specifically, β and σ are chosen so that the simulated wage flow equals its normalized initial level, $w_0 = 1$, consistent with a stationary wage in equilibrium. The resulting calibrated values are $\beta = 0.01$ and $\sigma = 0.194$. Sensitivity analyses indicate that alternative parametrization do not qualitatively affect either the calibration results or the dynamic properties of wealth and income distributions.

Appendix F

F1. This appendix demonstrates that an increase in patent breadth reduces both the quality-adjusted firm size, denoted by x_t, and the wage flow, w_t. Moreover, the effect of an increase in patent breadth on the expected stock market value of a monopolistic firm is analyzed.

To analyze the effect of patent breadth on the quality-adjusted firm size x_t, we focus on the inverse of the quality-adjusted firm size, i.e., $1/x_t$, as it proves analytically more tractable and facilitates comparative statics. From condition (C2) the following expression of the inverse of quality-adjusted firm size is obtained:

$$\frac{1}{x_t} = \frac{\beta}{\sigma} \frac{1}{(\mu)^{\frac{1}{1-\theta}}} \left[\frac{(\mu-1)}{\beta} \left(1 - \frac{\alpha}{\xi} z_t \right) - \xi - z_t \right], \qquad \text{(F1.1)}$$

from which the following expression is derived

$$\begin{aligned} \frac{\partial(1/x_t)}{\partial\mu} = -\frac{\beta}{\sigma} \frac{\frac{1}{1-\theta}(\mu)^{\frac{1}{1-\theta}-1}}{\left[(\mu)^{\frac{1}{1-\theta}}\right]^2} \left[\frac{(\mu-1)}{\beta} \left(1 - \frac{\alpha}{\xi} z_t\right) - (\xi - z_t) \right] + \\ + \frac{\beta}{\sigma} \frac{\frac{1}{\beta}\left(1 - \frac{\alpha}{\xi} z_t\right)}{(\mu)^{\frac{1}{1-\theta}}}. \end{aligned} \qquad \text{(F1.2)}$$

Because of the parameter rustications of the Appendix B, i.e., both inequalities $1 - z_t(\alpha/\rho) > 0$ and $(\rho - z_t) > 0$ hold. Condition (F1.2) implies that inequality $\partial(1/x_t)/\partial\mu > 0$ holds when

$$\begin{aligned} \frac{\partial(1/x_t)}{\partial\mu} = \frac{\frac{1}{1-\theta}}{\mu} \left[\frac{(\mu-1)}{\beta} \left(1 - \frac{\alpha}{\xi} z_t\right) - (\xi - z_t) \right] + \\ + \frac{1}{\beta} \left(1 - \frac{\alpha}{\xi} z_t\right) > 0, \end{aligned} \qquad \text{(F1.3)}$$

holds. Inequality (F1.3) can be written as

$$\begin{aligned} \frac{\partial(1/x_t)}{\partial\mu} = \frac{1}{\beta} \left(1 - \frac{\alpha}{\xi} z_t\right) \left(1 - \frac{1}{1-\theta} \frac{\mu-1}{\mu}\right) + \\ + \frac{1}{1-\theta} \frac{1}{\mu} (\xi - z_t) > 0. \end{aligned} \qquad \text{(F1.4)}$$

Since both inequalities $1 - z_t (\alpha/\xi) > 0$ and $(\xi - z_t) > 0$ hold, inequality (F1.4) holds when

$$\left(1 - \frac{1}{1-\theta}\frac{\mu - 1}{\mu}\right) > 0. \tag{F1.5}$$

The assumption $\mu < 1/\theta$, that is assumed to hold, ensures that $1 - (1/(1-\theta))((\mu - 1)/\mu) > 0$. Therefore, $\partial (1/x_t)/\partial\mu > 0$ holds, which implies inequality $\partial x_t/\partial\mu < 0$. Considering the expression of the wage flow (B7)

$$w_t = \frac{\alpha}{\xi}(\mu - 1)\left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}}\frac{E_t}{N_t}, \tag{F1.6}$$

we can obtain how the wage flow responds to a larger patent breadth, with a constant productivity-adjusted firm size x_t, by calculating

$$\frac{\partial w_t}{\partial\mu} = \frac{\alpha}{\xi}\left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}}\frac{E_t}{N_t}\left(1 - \frac{\mu - 1}{(1-\theta)\mu}\right) > 0, \tag{F1.7}$$

where inequality $\partial w_t/\partial\mu > 0$ follows because inequality

$1-(\mu - 1)/(1-\theta)\mu > 0$ holds. However, a larger patent breadth negatively affects productivity-adjusted firm size, as shown in equation (F1.4). Taking this effect into account, the response of the wage flow to an increase in patent breadth is given by

$$\frac{\partial w_t}{\partial\mu} = \frac{\alpha}{\xi}\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}\left[x_t\left(1 - \frac{\mu - 1}{(1-\theta)\mu}\right) + (\mu - 1)\frac{\partial x_t}{\partial\mu}\right], \tag{F1.8}$$

which is of indeterminate sign. The first term inside the brackets is positive, since $1-(\mu - 1)/(1-\theta)\mu > 0$, whereas the second term is negative, as $(\mu - 1)\,\partial x_t/\partial\mu < 0$. This establishes the result. Q.E.D.

The effect of an increase in patent breadth on the expected stock market value of a monopolistic firm is

$$\frac{\partial V_t}{\partial\mu} = \frac{[\exp(\xi u - \delta - \xi)\bullet t]}{\rho - (\xi u - \delta)} \times \left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} \left[\left(1 - \frac{\mu-1}{(1-\theta)\mu}\right)x_t + (\mu - 1)\frac{\partial x_t}{\partial\mu} - \frac{\partial w_t}{\partial\mu}\frac{L_0(i)}{\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}}\right]. \tag{F1.9}$$

Considering (F1.8), equation (F1.9) becomes

$$\frac{\partial V_t}{\partial \mu}=\frac{[\exp(\xi u-\delta-\xi)\bullet t]}{\xi-(\xi u-\delta)}\times\left[1-\frac{\alpha}{\xi}L_0\left(i\right)\right]$$
$$\left[\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}\left(1-\frac{\mu-1}{(1-\theta)\mu}\right)x_t+(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}\frac{\partial x_t}{\partial \mu}\right]. \tag{F1.10}$$

Q.E.D.

F2. This appendix demonstrates that an increase in the updating time per unit of human capital, u increases both the quality-adjusted firm size, denoted by x_t, and the wage flow, w_t. Moreover, the effect of an increase in the updating time per unit of human capital on the expected stock market value of a monopolistic firm is analyzed.

To analyze the effect of the updating time on the quality-adjusted firm size x_t, we consider condition (C2) form which expression is derived

$$\frac{\partial x_t}{\partial u}=\frac{\sigma}{\beta}\left(\mu\right)^{\frac{1}{1-\theta}}\frac{-\left(1-\frac{\alpha}{\xi}\frac{\mu-1}{\beta}\right)}{\left[\frac{(\mu-1)}{\beta}+z_t\left(1-\frac{\alpha}{\xi}\frac{\mu-1}{\beta}\right)-\rho\right]^2}, \tag{F2.1}$$

which is positive when $\mu>1+\left(\beta\xi\right)/\alpha$.

Considering the expression of the wage flow (B7)

$$w_t=\frac{\alpha}{\xi}\left(\mu-1\right)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}x_t, \tag{F2.2}$$

we can obtain how the wage flow responds to a larger patent breadth by calculating

$$\frac{\partial w_t}{\partial u}=\frac{\alpha}{\xi}\left(\mu-1\right)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}\frac{\partial x_t}{\partial u}, \tag{F2.3}$$

which is positive when $\mu>1+\left(\beta\xi\right)/\alpha$.

The effect of an increase in patent breadth on the expected stock market value of a monopolistic firm is

$$\frac{\partial V_t}{\partial u}=\left[\exp\left(\xi u-\delta-\xi\right)\bullet t\right]\xi t\times\Omega+\left[\exp\left(\xi u-\delta-\xi\right)\bullet t\right]\times$$
$$\times\frac{\left[(\mu-1)\left(\frac{1}{\mu}\right)^{(1/(1-\theta))}\frac{\partial x_t}{\partial u}Z_0-\frac{\partial w_t}{\partial u}L_0(i)\right][\xi-(\xi u-\delta)]+\xi\Omega}{[\xi-(\xi u-\delta)]^2}, \tag{F2.4}$$

where $\Omega = \frac{\left[\left((\mu-1)\left(\frac{1}{\mu}\right)^{(1/(1-\theta))}x_t-\sigma\right)Z_0-w_tL_0(i)\right]}{\xi-(\xi u-\delta)}$. Since $\Omega > 0$, $\partial x_t/du > 0$, and $\partial w_t/du > 0$, we have $\partial V_t/du > 0$ when the following parameter restriction $L_0(i)/Z_0 < \xi/\alpha$ holds.

Appendix G

G1. This Appendix obtains the optimal choices of a typical household κ assuming that the size of each household grows at a constant and common growth rate $n > 0$, that also represents the population growth rate. In this case, the utility function, the asset accumulation equation, the human-capital embodied time constraint, and the law of motion for human capital per capita of a household respectively are

$$\begin{gathered} U(\kappa) = \int_0^\infty e^{-(\rho-n)t}\ln(c_t(\kappa))\,dt, \\ \dot{a}_t(\kappa) = (r_t-n)\,a_t(\kappa) + w_t l_t(\kappa) - c_t(\kappa) + \psi(\kappa)\,\tau w_t L_t, \\ h_t(\kappa) = l_t(\kappa) + uh_t \\ \dot{h}_t(\kappa) = \xi(\kappa)\,uh_t - (\delta(\kappa)+n)\,h_t. \end{gathered} \tag{G1.1}$$

Because newborns are uneducated, population growth operates like depreciation of human capital per capita. The Hamiltonian is

$$\begin{gathered} H_t = e^{-(\rho-n)t}\ln c_t(\kappa) + \\ +\zeta_{at}\left[(r_t-n)\,a_t(\kappa) + w_t l_t(\kappa) - c_t(\kappa) + \psi(\kappa)\,\tau w_t L_t\right] + \\ +\zeta_{ht}\left[\xi(\kappa)\left(h_t(\kappa)-l_t(\kappa)\right) - (\delta(\kappa)+n)\,h_t\right], \end{gathered} \tag{G1.2}$$

where ζ_{at} and ζ_{ht} are the costate variables for the asset accumulation equation and the human capital accumulation equation respectively. The optimal conditions are:

$$\frac{\partial H_t}{\partial c_t(\kappa)} = 0 \Rightarrow \frac{e^{-(\rho-n)t}}{c_t(\kappa)} = \zeta_{at}, \tag{G1.3}$$

$$\frac{\partial H_t}{\partial l_t(\kappa)} = 0 \Rightarrow \zeta_{at}w_t - \zeta_{ht}\xi(\kappa) = 0, \tag{G1.4}$$

$$-\frac{\partial H_t}{\partial a_t(\kappa)} = \dot{\zeta}_{at} = -\zeta_{at}(r_t-n), \tag{G1.5}$$

$$-\frac{\partial H_t}{\partial h_t(\kappa)} = \dot{\zeta}_{ht} = \zeta_{ht}\left(\delta(\kappa)+n-\xi(\kappa)\right). \tag{G1.6}$$

From conditions (G1.5) and (G1.6), the law of motion of the costate variables ζ_{at} and ζ_{ht} are respectively $\left(\dot{\zeta}_{at}/\zeta_{at}\right) = n - r_t$ and $\left(\dot{\zeta}_{ht}/\zeta_{ht}\right) = \delta(\kappa) + n - \xi(\kappa)$. From condition (G1.4) we obtain $\left(\dot{\zeta}_{at}/\dot{\zeta}_{at}\right) + (\dot{w}_t/w_t) = \left(\dot{\zeta}_{ht}/\zeta_{ht}\right)$, that, using the law of motion of the costate variables ζ_{at} and ζ_{ht} can be written as: $(\dot{w}_t/w_t) = r_t + \delta(\kappa) - \xi(\kappa)$. From condition (G1.3), and using again the law of motion of the costate variables ζ_{at}, the standard law of motion of the per capita consumption of household κ becomes $\dot{c}_t(\kappa)/c_t(\kappa) = \dot{c}_t/c_t = r_t - \rho$. Using the law of motion of human capital of a household κ (6), the aggregate human capital accumulation of the economy is

$$\begin{aligned} \dot{h}_t &= \textstyle\int_0^1 \dot{h}_t(\kappa)\, dF(\kappa) = \\ &= h_t \textstyle\int_0^1 \left(\xi(\kappa) u - \delta(\kappa) - n\right) dF(\kappa) = h_t\left(\xi u - \delta - n\right), \end{aligned} \tag{G1.7}$$

where $\xi = \int_0^1 \xi(\kappa)\, dF(\kappa)$, $\delta = \int_0^1 \delta(\kappa)\, dF(\kappa)$. Therefore, the growth rate of human capital becomes:

$$g_h \equiv \frac{\dot{h}_t}{h_t} = \xi u - (\delta + n). \tag{G1.8}$$

Q.E.D.

G2. This Appendix derives the wage flow, w_t, and the quality-adjusted firm size, x_t, under growing population hypothesis. The law of motion of the wage flow with growing population, i.e., $(\dot{w}_t/w_t) = r_t + \delta(\kappa) - \xi(\kappa)$, generates a different wage equation. To determine the new equation of the wage flow, the same analysis as in Appendix B applies, with the only difference of the optimal condition (B4), that using $r_t = (\dot{w}_t/w_t) + \xi(\kappa) - \delta(\kappa)$ to substitute for r_t, can be written as:

$$\begin{aligned} \alpha &\left[(\mu - 1)\left(\tfrac{\theta}{\mu}\right)^{\frac{1}{1-\theta}} \tfrac{E_t}{N_t}\right] Z_t^{\alpha-1}(i)\, Z_t^{1-\alpha} = \\ &= \left(\tfrac{\dot{w}_t}{w_t} + \xi(\kappa) - \delta(\kappa)\right) w_t - \dot{w}_t. \end{aligned} \tag{G2.1}$$

Assuming symmetry, i.e. $Z_t(i) = Z_t$, $\xi(\kappa) = \xi$, and $\delta(\kappa) = \delta$, equation (G2.1) reduces to

$$\alpha\left[(\mu - 1)\left(\frac{\theta}{\mu}\right)^{\frac{1}{1-\theta}} \frac{E_t}{N_t}\right] = (\xi - \delta)\, w_t. \tag{G2.2}$$

From (G2.2), we can derive the wage flow:

$$w_t = \frac{\alpha}{\xi - \delta}\left[(\mu - 1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} x_t\right] \tag{G2.3}$$

where in the last equality on the right hand side (18) has been used.

Using the growth rate of aggregate quality, $z_t = \left(\dot{Z}_t / Z_t\right)$, and substituting equation (G2.3) for the wage flow w_t, together with the condition $(\dot{x}_t/x_t) = (\dot{w}_t/w_t) = r_t + \delta - \xi$, equation (22) can be rewritten as:

$$\begin{aligned} r_t = & \frac{(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t - \sigma Z_t}{\beta\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t} + \\ & - \frac{\frac{\alpha}{n}\left[(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} x_t\right] L_t}{\beta\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} Z_t x_t} + z_t + r_t - (\xi - \delta)\,. \end{aligned} \tag{G2.4}$$

Using $\dot{Z}_t = L_t$, the expression for the quality-adjusted firm size x_t, as given in condition (G2.4), can be equivalently rewritten as:

$$\begin{aligned} x_t = & \frac{\sigma}{\beta\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}}\left[\frac{\mu-1}{\beta} + z_t\left(1 - \frac{\alpha}{(\xi-\delta)}\frac{\mu-1}{\beta}\right) - (\xi - \delta)\right]^{-1} = \\ = & \frac{\sigma}{\beta}\,(\mu)^{\frac{1}{1-\theta}}\left[\frac{(\mu-1)}{\beta} + z_t\left(1 - \frac{\alpha}{(\xi-\delta)}\frac{\mu-1}{\beta}\right) - (\xi - \delta)\right]^{-1}. \end{aligned} \tag{G2.5}$$

Equation (G2.5) implies the stationarity of the productivity-adjusted firm size and the of the wage flow, i.e., $(\dot{x}_t/x_t) = (\dot{w}_t/w_t) = r_t + \delta - \xi = 0$, which pins down the interest rate $r_t = \xi - \delta$. Q.E.D.

G3. This Appendix derives the monopolistic firm stock market value, V_t, the growth rate of the financial asset per capita, $\dot{a}_t/a_t$, and the growth rate of final output per capita, $\dot{y}_t/y_t$, where $y_t = Y_t/L_t$, in the case of a constant population growth rate $n > 0$. Here, we rewrite the monopolistic firm stock market value (29)

$$V_t = \int_t^{\infty} e^{-r_s \bullet s}\left(\Pi_s - w_s L_s(i)\right) ds, \tag{G3.1}$$

that, following the same steps as in the previous analysis and using

$z_t = g_h + n = \xi u - (\delta + n) + n = (\xi u - \delta)$, simplifies to:

$$V_t = \exp(\xi u - \xi)\, t \times \\ \times \frac{\left[\left((\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} x_t - \sigma\right) Z_0 - w_t L_0(i)\right]}{\xi - \xi u}, \tag{G3.2}$$

which is strictly positive because $\xi - \xi u > 0$ holds. The growth rate of the financial assets per capita is $\dot{a}_t/a_t - n = \dot{V}_t/V_t + \dot{N}_t/N_t - n = z_t + \dot{N}_t/N_t - n$. Using (11) and assuming symmetry, the aggregate final output is $Y_t = (\theta/\mu)^{\frac{\theta}{1-\theta}} Z_t E_t$. Because the quality-adjusted firm size (x_t) is constant, we have $\dot{E}_t/E_t = \dot{N}_t/N_t$ and the growth rate of final output per capita is: $g_y \equiv \dot{Y}_t/Y_t - n = z_t + \dot{N}_t/N_t - n$.

Using the law of motion of the number of monopolistic firms (34), the law of motion of the number of monopolistic firms (34) generates the growth rate of intermediate goods $\dot{N}_t/N_t = (g_h + n)/(1 + \phi)$. Therefore, the growth rate of the financial assets per capita and of the final output per capita become:

$$\frac{\dot{a}_t}{a_t} - n = \frac{\dot{Y}_t}{Y_t} - n = z_t + \frac{g_h + n}{1 + \phi} - n = \left(\frac{2 + \phi}{1 + \phi}\right)(\xi u - \delta) - n, \tag{G3.3}$$

where, in the last equality, we have used $z_t = g_h + n$ and $g_h = \xi u - (\delta + n)$. Q.E.D.

In examining the dynamics of wealth and income distributions, our analysis has focused on their evolution across households over time. However, in the context of a growing population, it is more appropriate to consider the distributional dynamics across dynastic households - that is, extended family units whose size evolves at the population growth rate n. Accordingly, we analyze the evolution of wealth and income distributions by tracking the trajectories of dynastic households indexed by κ, each growing at rate n. Under this framework, the fundamental structure governing the dynamics of consumption, wealth, and income distributions - as well as the construction of inequality measures such as the Gini coefficient - remains formally equivalent to the case with a stationary population. Nevertheless, a key analytical adjustment is required: where relevant, the interest rate r_t should be replaced by the population growth rate n. This substitution stems from the fact that, under balanced growth with population expansion, the normalized variables satisfy $(\dot{x}_t/x_t) = (\dot{w}_t/w_t) = r_t - (\xi - \delta) = 0$, implying $r_t = \xi - \delta$. Q.E.D.

Appendix H

This appendix establishes the robustness of the model to heterogeneous degrees of patent protection across industries.

With industry-specific patent protection, equilibrium markups may differ across varieties. To preserve consistency with the baseline analysis and following the same steps as in Appendix B, we assume that the condition $\mu_i < 1/\theta$ holds for every variety i. Under this restriction, profit maximization implies that the price charged by intermediate firm i is constant over time and equal to $p_t(i) = \mu_i$. The demand for each intermediate variety $X_t(i)$, the associated profit flow $\Pi_t(i)$, the law of motion for variety-specific quality $Z_t(i)$, and the expected stock market value $V_t(i)$ are given, *mutatis mutandis*, by equations (11), (13), (12), and (14), respectively. Repeating the steps of Appendix B, the wage flow can be expressed as

$$w_t = \frac{\alpha}{\xi(\kappa)}(\mu_i - 1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}\left(\frac{Z_t}{Z_t(i)}\right)^{1-\alpha} x_t(i), \tag{H1}$$

where the productivity-adjusted firm size is defined as

$$\begin{aligned} x_t(i) &\equiv \tfrac{X_t(i)}{Z_t^{\alpha}(i)Z_t^{1-\alpha}}(\mu_i)^{\frac{1}{1-\theta}} = \\ &= \tfrac{X_t(i)/Z_t(i)}{(Z_t/Z_t(i))^{1-\alpha}}(\mu_i)^{\frac{1}{1-\theta}} = (\theta)^{\frac{1}{1-\theta}}\tfrac{E_t}{N_t}, \end{aligned} \tag{H2}$$

The final expression shows that $x_t(i)$ is homogeneous across varieties, i.e., $x_t(i) = x_t$, for each i. Moreover, from the R&D condition (12), $\dot{Z}_t(i)/\,Z_t(i) = g_h$, implying that the ratio $Z_t/\,Z_t(i)$ is constant over time. Under perfect capital markets, the asset-pricing condition for variety i is

$$r_t = \frac{\Pi_t(i) - w_t L_t(i)}{V_t(i)} + \frac{\dot{V}_t(i)}{V_t(i)}. \tag{H3}$$

Following the analysis of Section 3.4, considering the free entry condition for the creation of new varieties $V_t(i) = \beta X_t(i) = \beta(1/\mu_i)^{\frac{1}{1-\theta}} Z_t (Z_t(i)/Z_t)^{\alpha} x_t$, using the growth rate of aggregate quality, z_t, and substituting equation (H1) for the wage flow w_t, together with the condition $(\dot{x}_t/x_t) = r_t - \xi(\kappa)$, equation (H3) can be rewritten as:

$$r_t = \frac{(\mu_i - 1)}{\beta} - \frac{\sigma}{\beta\left(\frac{1}{\mu_i}\right)^{\frac{1}{1-\theta}} (Z_t(i)/Z_t)^{\alpha} x_t} + \\ - \frac{\frac{\alpha}{\beta\xi(\kappa)}(\mu_i - 1) L_t(i)}{Z_t(i)} + z_t + r_t - \xi(\kappa), \tag{H4}$$

where $Z_t(i)/Z_t$ is constant. Assuming the normalization $Z_0/\ Z_0(i) = 1$, and using $\dot{Z}_t(i) = L_t(i)$, we obtain

$$x_t = \frac{\frac{\sigma}{\beta}\mu_i^{\frac{1}{1-\theta}}}{\left[\frac{\mu_i - 1}{\beta} + z_t(i)\left(1 - \frac{\alpha}{\xi(\kappa)}\frac{\mu - 1}{\beta}\right) - \xi(\kappa)\right]}, \tag{H5}$$

where equation (12) implies the equality $z_t(i) = z_t$ for each i. Since $z_t(i) = z_t = g_h = \xi u - \delta$ is constant, it follows that the productivity-adjusted firm size x_t is stationary. From the wage equation, this implies $\dot{w}_t/w_t = r_t - \xi(\kappa) = 0$, so that $r_t = \xi(\kappa) = \xi$. The Euler equation for consumption then implies $\dot{c}_t(\kappa)/c_t(\kappa) = \dot{c}_t/c_t = r_t - \rho = \xi - \rho$. Consequently, the wage flow derived under heterogeneous patent protection (H1) coincides with that in the baseline model (B7).

Finally, stationarity of x_t implies that the ratio $X_t(i)/Z_t(i)$ is also stationary. Hence $\dot{X}_t(i)/X_t(i) = \dot{Z}_t(i)/Z_t(i) = \dot{Z}_t/Z_t = g_h = z_t$.

Following the same calculations of the main text, the monopolistic firm stock market value of a variety i becomes:

$$V_t(i) = [\exp(\xi u - \delta - \xi) \bullet t] \times \\ \times \frac{\left[\left((\mu_t - 1)\left(\frac{1}{\mu_i}\right)^{\frac{1}{1-\theta}} x_t \left(\frac{Z_t(i)}{Z_t}\right)^{\alpha} - \sigma\right) Z_0 - w_t L_0(i)\right]}{\xi - (\xi u - \delta)}, \tag{H6}$$

where $Z_t(i)/\ Z_t$ is constant. Q.E.D.

We now show that the growth rates of aggregate asset value and GDP per capita coincide with those obtained in the symmetric benchmark. The aggregate asset value is given by $a_t = \int_0^{N_t} V_t(i)\, di$. Its growth rate can be decomposed into the growth of firm value and the growth in the number of varieties:

$$\frac{\dot{a}_t}{a_t} = \frac{\dot{V}_t(i)}{V_t(i)} + \frac{\dot{N}_t}{N_t} = z_t + \frac{\dot{N}_t}{N_t} = \frac{2 + \phi}{1 + \phi} g_h. \tag{H7}$$

Using equation (11), the aggregate final output can be written as

$$Y_t = (\theta)^{\frac{\theta}{1-\theta}} \frac{E_t}{N_t} Z_t^{1-\alpha} \int_0^{N_t} \left(\frac{1}{\mu_i}\right)^{\frac{\theta}{1-\theta}} Z_t^{\alpha}(i)\, di, \tag{H8}$$

which can be rewritten as

$$\begin{aligned} Y_t = (\theta)^{\frac{\theta}{1-\theta}} \tfrac{E_t}{N_t} Z_0^{1-\alpha} e^{(1-\alpha)g_h t} \int_0^{N_t} \left(\tfrac{1}{\mu_i}\right)^{\frac{\theta}{1-\theta}} Z_0^{\alpha}(i)\, e^{\alpha g_h t} di = \\ = (\theta)^{\frac{\theta}{1-\theta}} \tfrac{E_t}{N_t} Z_0^{1-\alpha} e^{g_h t} \int_0^{N_t} \left(\tfrac{1}{\mu_i}\right)^{\frac{\theta}{1-\theta}} Z_0^{\alpha}(i)\, di. \end{aligned} \tag{H9}$$

The growth rate of equation (H9) is

$$\frac{\dot{Y}_t}{Y_t} = g_h + \frac{g_h}{1+\phi} = \left(\frac{2+\phi}{1+\phi}\right)(\xi u - \delta), \tag{H10}$$

where we have used $g_h = \xi u - \delta$. Q.E.D.

Appendix I

This Appendix obtains the optimal choices of a typical household κ. Each consumer maximizes eq. (1) choosing the optimal plan of $c_t(\kappa)$, $l_t(\kappa)$, subject to the law of motion of the asset equation and human capital as in equations (50) and (6) respectively, and to the time constraint as in eq. (5). The current value Hamiltonian is

$$\begin{gathered} H_t = \ln(c_t(\kappa)) + \\ +\lambda_{at}\left[r_t a_t(\kappa) + (1-\tau) w_t (l_t(\kappa) + u h_t) + \psi(\kappa)\tau w_t h_t - c_t(\kappa)\right] + \\ +\lambda_{ht}\left[\xi(\kappa)(h_t(\kappa) - l_t(\kappa)) - \delta(\kappa) h_t\right]. \end{gathered} \tag{I1}$$

The optimal conditions are:

$$\frac{\partial H_t}{\partial c_t(\kappa)} = 0 \Rightarrow \frac{1}{c_t(\kappa)} = \lambda_{at}, \tag{I2}$$

$$\frac{\partial H_t}{\partial l_t(\kappa)} = 0 \Rightarrow \lambda_{at}(1-\tau) w_t - \lambda_{ht}\xi(\kappa) = 0, \tag{I3}$$

$$-\frac{\partial H_t}{\partial a_t(\kappa)} = \dot{\lambda}_{at} - \rho\lambda_{at} = -\lambda_{at} r_t, \tag{I4}$$

$$-\frac{\partial H_t}{\partial h_t(\kappa)} = \dot{\lambda}_{ht} - \rho\lambda_{ht} = -\lambda_{ht}\xi(\kappa) - \lambda_{at}(1-\tau) w_t. \tag{I5}$$

From conditions (I4), (I5), and (I3) the law of motion of the costate variables λ_{at} and λ_{ht} are respectively $\left(\dot{\lambda}_{at}/\lambda_{at}\right) = \rho - r_t$ and $\left(\dot{\lambda}_{ht}/\lambda_{ht}\right) = \rho - 2\xi(\kappa)$. From condition (I3) we obtain $\left(\dot{\lambda}_{at}/\lambda_{at}\right) + (\dot{w}_t/w_t) = \left(\dot{\lambda}_{ht}/\lambda_{ht}\right)$, that, using the law of motion of the costate variables λ_{at}, i.e., $\left(\dot{\lambda}_{at}/\lambda_{at}\right) = \rho - r_t$, and the law of motion of the costate variables λ_{ht} , i.e., $\left(\dot{\lambda}_{ht}/\lambda_{ht}\right) = \rho - 2\xi(\kappa)$, can be written as: $(\dot{w}_t/w_t) = r_t - 2\xi(\kappa)$. Using again the law of motion of the costate variables λ_{at}, from condition (I2), the standard law of motion of the per capita consumption becomes $\dot{c}_t(\kappa)/c_t(\kappa) = r_t - \rho$.

Using the law of motion of human capital of a household κ (6), the aggregate human capital accumulation of the economy is

$$\begin{aligned} \dot{h}_t &= \textstyle\int_0^1 \dot{h}_t(\kappa)\, dF(\kappa) = \\ &= h_t \textstyle\int_0^1 (\xi(\kappa) u - \delta(\kappa))\, dF(\kappa) = h_t(\xi u - \delta), \end{aligned} \tag{I6}$$

where $\xi = \int_0^1 \xi(\kappa)\, dF(\kappa)$, $\delta = \int_0^1 \delta(\kappa)\, dF(\kappa)$. Therefore, the growth rate of human capital becomes:

$$g_h \equiv \frac{\dot{h}_t}{h_t} = \xi u - \delta. \tag{I7}$$

Q.E.D.

Following the same steps of the previous Appendixes, we are able to fully determine all the variables of this version of the model. All the detailed calculations can be replicated by the reader following the same steps of the previous Appendixes and are disposable upon request to the author. To save on space, here we report the variables useful for the analysis of the main text.

Considering the growth rate of the aggregate human capital of the economy, condition $\dot{Z}_t(i) = h_t(i)$ implies the growth rate of aggregate quality, $z_t = g_h = (\xi u - \delta)$, is stationary. The quality-adjusted firm size x_t is stationary and equal to

$$x_t = \frac{\sigma}{\beta}(\mu)^{\frac{1}{1-\theta}} \left[\frac{(\mu-1)}{\beta} + (\xi u - \delta)\left(1 - \frac{\alpha}{2\xi}\frac{\mu-1}{\beta}\right) - 2\xi\right]^{-1}, \tag{I8}$$

and the wage flow is also stationary and equal to

$$w_t = \frac{\alpha}{2\xi}\left[(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} x_t\right]. \quad \text{(I9)}$$

Because condition $(\dot{w}_t/w_t) = r_t - 2\xi = 0$ holds, the interest rate is also stationary and equal to $r_t = 2\xi$. The monopolistic firm stock market value becomes:

$$V_t = [\exp(\xi u - \delta - 2\xi) \bullet t] \times \\ \times \frac{\left[\left((\mu-1)\left(\frac{1}{\mu}\right)^{(1/(1-\theta))} x_t - \sigma\right) Z_0 - w_t h_0(i)\right]}{2\xi - (\xi u - \delta)}, \quad \text{(I10)}$$

where $h_s(i)$ indicates the the human-capital embodied labor employed by the monopolistic firm i, and inequality $2\xi - (\xi u - \delta) = \xi(2-u) + \delta > 0$ always holds. The quality-adjusted firm size x_t and the wage flow per unit of human capital w_t are constants over time, while the aggregate technology Z_t and the aggregate (and average) human-capital embodied labor h_t grow at the common rate $z_t = g_h = (\xi u - \delta)$. where $2\xi - (\xi u - \delta) = \xi(2-u) + \delta > 0$ always holds.

The growth rate of the financial asset per capita and of final output

$$\frac{\dot{a}_t}{a_t} = \frac{\dot{Y}_t}{Y_t} = z_t + \frac{g_h}{1+\phi} = \left(\frac{2+\phi}{1+\phi}\right)(\xi u - \delta), \quad \text{(I11)}$$

where we have used $z_t = g_h = \xi u - \delta$. Q.E.D.

Appendix L

In this Appendix, we show the dynamics of the share of human capital and of wealth distribution. The dynamic law of the share of human capital of a household κ is

$$\dot{s}_{h,t}(\kappa) = \xi(\kappa) u - \delta(\kappa) - (\xi u - \delta) s_{h,t}(\kappa). \quad \text{(L1)}$$

The differential equation (L1) admits a unique solution that describes the time trajectory of the share of human capital of a household κ as

$$s_{h,t}(\kappa) = \frac{\xi(\kappa)u - \delta(\kappa)}{\xi u - \delta} + \\ - e^{-(\xi u - \delta)t}\left(\frac{\xi(\kappa)u - \delta(\kappa)}{(\xi u - \delta)} - s_{h,0}(\kappa)\right). \quad \text{(L2)}$$

The time path of household κ's share of human capital increases over time if its initial share is sufficiently large, that is, if $s_{h,0}(\kappa) >$

$[\xi(\kappa)u - \delta(\kappa)] / (\xi u - \delta)$. Otherwise, the household's share of human capital declines over time.

The growth rate of share of wealth owned by household κ, $s_{a,t}(\kappa)$, is given by

$$\begin{aligned} \dot{s}_{a,t}(\kappa) = \tfrac{c_t}{a_t}\left(s_{a,t}(\kappa) - s_{c,t}(\kappa)\right) + \\ -\tfrac{w_t h_t}{a_t}\left[\left(s_{a,t}(\kappa) - \psi(\kappa)\tau\right) - (1-\tau)s_{h,t}(\kappa)\right], \end{aligned} \tag{L3}$$

where $a_t = a_0 e^{\left[z_t + \frac{g_h}{1+\phi}\right]*t}$, $h_t = h_0 e^{g_h t}$, $g_h = z_t = (\xi u - \delta)$, $c_t = c_0$, w_t is constant. Since the functions on the right-hand side of the differential equation (L3) are continuous, standard results guarantee the existence and uniqueness of a solution to (43) for any given initial condition (see, e.g., Boyce et al., 2021).

Let $s_{I,t}(\kappa) \equiv I_t(\kappa)/I_t$ denote the share of income received by household κ. Then, we have

$$\begin{aligned} s_{I,t}(\kappa) = \tfrac{r_t a_t(\kappa) + (1-\tau)w_t h_t(\kappa) + \psi(\kappa)\tau w_t h_t}{r_t a_t + w_t h_t} = \\ = \tfrac{r_t a_t}{r_t a_t + w_t h_t} s_{a,t}(\kappa) + \tfrac{w_t h_t\left[(1-\tau)s_{h,t}(\kappa) + \psi(\kappa)\tau\right]}{r_t a_t + w_t h_t}, \end{aligned} \tag{L4}$$

where $\int_0^1 \psi(\kappa)\,d\kappa = 1$, and $h_t = \int_0^1 h_t(\kappa)\,d\kappa$. Equation (L4) determines the evolution of the share of income received by household κ and allows us to derive any moment of the income distribution

$$\sigma_{I,t} = \frac{r_t a_t}{r_t a_t + w_t h_t}\sigma_{a,t} + \frac{2}{3}\frac{w_t h_t}{r_t a_t + w_t h_t}, \tag{L5}$$

where $r_t = 2\xi$, and $\sigma_{a,t}$ represents the Gini coefficient of wealth. Q.E.D.

Appendix M

This appendix indicates that an increase in patent breadth reduces both the quality-adjusted firm size, denoted by x_t. Moreover, the effect of an increase in patent breadth on the expected stock market value of a monopolistic firm is analyzed.

To analyze the effect of patent breadth on the quality-adjusted firm size x_t, we focus on the inverse of the quality-adjusted firm size, i.e., $1/x_t$, as it proves analytically more tractable and facilitates comparative statics. From condition (I8) the following expression of the inverse of quality-adjusted firm size is obtained:

$$\frac{1}{x_t} = \frac{\beta}{\sigma}\frac{1}{(\mu)^{\frac{1}{1-\theta}}}\left[\frac{(\mu-1)}{\beta}\left(1 - \frac{\alpha}{\rho}z_t\right) - 2\xi - z_t\right], \tag{M1}$$

from which the following expression is derived

$$\begin{aligned}\frac{\partial(1/x_t)}{\partial\mu} = \tfrac{1}{\beta}\left(1-\tfrac{\alpha}{\xi}z_t\right)\left(1-\tfrac{1}{1-\theta}\tfrac{\mu-1}{\mu}\right)+ \\ +\tfrac{1}{1-\theta}\tfrac{1}{\mu}\left(\xi-z_t\right)>0.\end{aligned} \tag{M2}$$

Since both inequalities $1-z_t\,(\alpha/\rho)>0$ and $(\xi-z_t)>0$ hold, inequality (M2) holds when $1-(1/\,(1-\theta))\,((\mu-1)\,/\mu)>0$, which always holds because inequality $\mu<1/\theta$ is assumed to hold. Therefore, $\partial\,(1/x_t)\,/\partial\mu>0$ holds, which implies inequality $\partial x_t/\partial\mu<0$. Considering the expression of the wage flow (I9), the response of the wage flow to an increase in patent breadth is given by

$$\frac{\partial w_t}{\partial\mu}=\frac{\alpha}{2\xi}\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}\left[x_t\left(1-\frac{1}{1-\theta}\frac{\mu-1}{\mu}\right)+(\mu-1)\frac{\partial x_t}{\partial\mu}\right], \tag{M3}$$

which is of indeterminate sign. The first term inside the brackets is positive, since$1-(1/\,(1-\theta))\,((\mu-1)\,/\mu)>0$, whereas the second term is negative, as $(\mu-1)\,\partial x_t/\partial\mu<0$. This establishes the result.

The effect of an increase in patent breadth on the expected stock market value of a monopolistic firm is

$$\begin{aligned}\frac{\partial V_t}{\partial\mu} = \frac{[\exp(\xi u-\delta-2\xi)\bullet t]}{2\xi-(\xi u-\delta)}\times\left[1-\tfrac{\alpha}{\rho}h_0\,(i)\right]\times \\ \times\left(\tfrac{1}{\mu}\right)^{\frac{1}{1-\theta}}\left[\left(1-\tfrac{1}{1-\theta}\tfrac{\mu-1}{\mu}\right)x_t+(\mu-1)\,\tfrac{\partial x_t}{\partial\mu}\right].\end{aligned} \tag{M4}$$

Q.E.D.

To analyze the effect of the updating time on the quality-adjusted firm size x_t, we consider condition (I8) form which expression is derived

$$\frac{\partial x_t}{\partial u}=\frac{\sigma}{\beta}\,(\mu)^{\frac{1}{1-\theta}}\frac{-\left(1-\frac{\alpha}{2\xi}\frac{\mu-1}{\beta}\right)}{\left[\frac{(\mu-1)}{\beta}+z_t\left(1-\frac{\alpha}{2\xi}\frac{\mu-1}{\beta}\right)-2\xi\right]^2}, \tag{M5}$$

which is positive when $\mu>1+(2\beta\xi)\,/\alpha$.

Considering the expression of the wage flow (I9), we can obtain how the wage flow responds to a larger patent breadth by calculating

$$\frac{\partial w_t}{\partial u}=\frac{\alpha}{2\xi}\,(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}}\frac{\partial x_t}{\partial u}, \tag{M6}$$

which is positive when $\mu > 1 + (2\beta\xi)/\alpha$.

The effect of an increase in patent breadth on the expected stock market value of a monopolistic firm is

$$\begin{aligned} \frac{\partial V_t}{\partial u} &= [\exp(\xi u - \delta - 2\xi) \bullet t]\, \xi t \times \Omega + \\ &+ [\exp(\xi u - \delta - 2\xi) \bullet t] \times \\ &\times \frac{\left[(\mu-1)\left(\frac{1}{\mu}\right)^{\frac{1}{1-\theta}} \frac{\partial x_t}{\partial u} Z_0 - \frac{\partial w_t}{\partial u} h_0(i)\right][2\xi-(\xi u-\delta)]+\xi\Omega}{[2\xi-(\xi u-\delta)]^2}, \end{aligned} \tag{M7}$$

where $\Omega = \frac{\left[\left((\mu-1)\left(\frac{1}{\mu}\right)^{(1/(1-\theta))} x_t - \sigma\right) Z_0 - w_t h_0(i)\right]}{2\xi-(\xi u-\delta)}$. Since $\Omega > 0$, $\partial x_t/du > 0$, and $\partial w_t/du > 0$, we have $\partial V_t/du > 0$ when the following parameter restriction $h_0(i)/Z_0 < (2\xi)/\alpha$ holds. Otherwise, when inequality $h_0(i)/Z_0 < (2\xi)/\alpha$ does not hold, the sign of condition (M7) is of indeterminate sign. Q.E.D.

Q.E.D.

Appendix N

This appendix provides an explanation of how the parameter values are calibrated when the firms remunerate labor and updating time. To save on space, here we indicate the value of parameters that differ from the baseline calibration values of quantitative of the main text detailed in Appendix E

The interest rate, denoted by r_t, is set to the conventional value of 0.0331. Since the model optimal conditions of a consumption imply $r_t = 2\xi$, the value of parameter ξ is set at $\xi = r_t/2 = 0.01655$. We calibrate the depreciation rate of human capital δ using the model-implied growth rate of human capital, $g_h = (\xi u - \delta)$, targeting its empirical counterpart. Following Mulligan and Sala-i-Martin (2000), we set the average annual growth rate of human capital in the U.S. economy to 0.4 percent. The value of the the depreciation rate of human capital is $\delta = 0$.

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ROMA TRE
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